PHYS 30672 MATHS METHODS Examples 1

1. Show that the following limit defines the Dirac δ function:

$$\lim_{a\downarrow 0} \frac{1}{\sqrt{\pi}a} \exp(-x^2/a^2)$$

Hint: See the mathematica notebook dirac.nb on the web site!

2. From the definition of a δ function and the Fourier transform show that

$$\int_{-\infty}^{\infty} f^*(x)g(x)dx = \int_{-\infty}^{\infty} \tilde{f}^*(k)\tilde{g}(k)dk.$$

3. Show that the integration operators L_1 and L_2 defined by

$$(L_1 f)(x) = \int_0^x f(x')dx',$$

$$(L_2 f)(x) = \int_0^1 G(x,x')f(x')dx',$$

are linear operators

4. If one were to define the adjoint of a linear operator L by

$$(\boldsymbol{e}_i\,,\,L^\dagger\,\boldsymbol{u}_k) = (\boldsymbol{u}_k\,,\,L\,\boldsymbol{e}_i)^* = (L\,\boldsymbol{e}_i\,,\,\boldsymbol{u}_k)$$

where $\boldsymbol{e}_i\,,\,\boldsymbol{u}_k$ are basis vectors in the domain and codomain respectively, show that this implies

$$(\boldsymbol{a}\,,\,L^\dagger\,\boldsymbol{b}) = (\boldsymbol{b}\,,\,L\,\boldsymbol{a})^* = (L\,\boldsymbol{a}\,,\,\boldsymbol{b})$$

where $\boldsymbol{a}\,,\,\boldsymbol{b}$ are arbitrary vectors in the domain and codomain, respectively.

5. Consider the generalized eigenvalue problem

$$L\,y(x) = \lambda\,\rho(x)\,y(x)$$

where L is an Hermitian differential operator, $\rho(x)$ is a real, positive definite weight function. Prove that the eigenvalues are real; and that eigenvectors $u(x)\,v(x)$ corresponding to different eigenvalues satisfy the orthogonality condition

$$\int dx\,\rho(x)\,u(x)^*\,v(x) = 0\quad .$$

6. Chebychev polynomials $T_n(x)$ of type one are polynomials of order n which satisfy the differential equation

$$(1-x^2)T_n''(x) - xT_n'(x) + n^2T_n(x) = 0\ . \qquad (1)$$

in the range $-1 < x < 1$. Multiply this equation by $(1-x^2)^{-1/2}$ and show that the resulting equation can be put into the form of a Sturm-Liouville equation. What is the form of the orthogonality relation for these functions?

For $n = 0$, equation (1) has the trivial solution $T_0(x) = 1$. Find a second solution in the form of an indefinite integral, and then evaluate the integral to obtain its explicit form.

7. Show that the eigenvalues λ of the Sturm-Liouville equation

$$(-py')' + qy = \lambda\rho y$$

are all positive if $p(x), q(x)$ and $\rho(x)$ are all positive for $a \le x \le b$ and $y(a) = y(b) = 0$. (All functions are real).

Hint: Start by multiplying the equation by y and integrate from a to b using integration by parts.

8. The function $u(x)$ is a solution of the Sturm-Liouville equation

$$-\frac{d}{dx}(p(x)\,u') + qu = \lambda\rho u, \qquad \text{with } p > 0, \rho > 0,$$

satisfying the boundary condition $u'/u = c$ at $x = a$. We also know that $u(x)$ has a zero at $x = X$.

If λ changes by a small amount to μ, a new solution $v(x)$ “close” to $u(x)$ is obtained satisfying the same boundary condition at $x = a$, with a corresponding zero at $x = Y$

Show that if $\mu > \lambda$ then $X > Y$.

Hint: Use the basic relation $\int_a^X dx\,(vLu - uLv) = [p(uv' - vu')]_a^X$ to show $u'v > 0$ at $x = X$, and consider its implications for the zeroes by sketching the behaviour of $u(x)$ and $v(x)$ close to $x = X$ for the cases $u'(X) > 0$, $u'(X) < 0$.