PHYS 30672 MATHS METHODS Examples 2

1. If $G(x, x')$ is the Green function for the Linear operator $\mathcal{L}$, what is the Green function $\overline{G}(x, x')$ corresponding to the linear operator $\overline{\mathcal{L}} = f(x)\mathcal{L}$?

2. Find the Green function $G(x, x')$ for the operator $\mathcal{L}y(x) \equiv \frac{d^2}{dx^2}y(x)$ in the range $0 \leq x \leq L$, where $y(0) = y(L) = 0$
(i) in the form of an eigenvalue expansion.
(ii) in the form of simple expressions for $x < x'$ and $x > x'$.

3. Find the Green function $G(x, x')$ for the operator

$$\mathcal{L}y(x) = \frac{d}{dx}\left(x\frac{d}{dx}y\right)$$

in the range $0 < x < 1$, where $y(0)$ is finite, and $y(1) = 0$, in the form as in 2.ii above.

4. The time dependent Schrödinger equation can be written in the form

$$i\hbar\frac{\partial\Psi(\boldsymbol{x}, t)}{\partial t} + \frac{\hbar^2\nabla^2\Psi(\boldsymbol{x}, t)}{2m} = V(\boldsymbol{x})\Psi(\boldsymbol{x}, t) \equiv \rho(\boldsymbol{x}, t). \qquad (1)$$

The Green function for the wave operator is defined by

$$\left[i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2\nabla^2}{2m}\right]G(\boldsymbol{x}, t; \boldsymbol{x}', t') = \delta^3(\boldsymbol{x} - \boldsymbol{x}')\delta(t - t') \,.$$

Use the Fourier transform technique to show that the Green function

$$G(\boldsymbol{x}, t) = G(\boldsymbol{x}, t; \boldsymbol{0}, 0)$$

satisfying the causal boundary condition $G(\boldsymbol{x}, t < 0) = 0$ is given by

$$G(\boldsymbol{x}, t) = -\frac{i}{\hbar}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\mathrm{e}^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega_k t)}$$

for $t > 0$, where $\hbar\omega_k = \dfrac{(\hbar)^2k^2}{2m}$.

For incoming particles scattering from a short range potential, one would expect

$$\Psi(\boldsymbol{x}, t) \to \Phi(\boldsymbol{x}, t) \qquad (2)$$

for both $t \to -\infty$ and for $V(\boldsymbol{x}) \to 0$, where $\Phi(\boldsymbol{x}, t)$ is a known "incoming" wavefunction satisfying

$$i\hbar\frac{\partial\Phi(\boldsymbol{x}, t)}{\partial t} + \frac{\hbar^2\nabla^2\Phi(\boldsymbol{x}, t)}{2m} = 0.$$

Write down the standard Green function solution for (1) to obtain an equation for $\Psi(\boldsymbol{x}, t)$ in terms of $G(\boldsymbol{x}, t; \boldsymbol{x}', t')$ and $V(\boldsymbol{x})$ and show that it satisfies the boundary conditions $\Psi(\boldsymbol{x}, t) \to 0$ for both $t \to -\infty$ and $V(\boldsymbol{x}) \to 0$.

Modify this to obtain an expression for $\Psi(\boldsymbol{x}, t)$ in terms of $G(\boldsymbol{x}, t; \boldsymbol{x}', t')$ and $\Phi(\boldsymbol{x}, t)$ which satisfies the boundary conditions (2) and is valid to first order in the potential .

5. The heat conduction equation for a long thin rod is

$$C\frac{\partial T}{\partial t} - K\frac{\partial^2 T}{\partial x^2} = W(x, t)$$

where $T(x, t)$ is temperature, C is heat capacity, K is conductivity and $W(x, t)$ is a variable heat source.

Find the Green function for the operator $\left(\frac{\partial}{\partial t} - D\frac{\partial^2}{\partial x^2}\right)$ where $D = \frac{k}{c}$, corresponding to $T \to 0$ as $|x| \to \infty$.

(i) Write down the differential equation for $G(x, x', t, t')$.
(ii) For $x' = 0, t' = 0$, show that the Fourier transform of $G(x, t)$ is

$$\tilde{G}(k, \omega) = \frac{i}{\omega + iDk^2}$$

(iii) Write down the expression for $G(x, t)$ in terms of an integral over k and ω, defined so that $G(x, t) = 0$ for $t < 0$.
(iv) Show that for $t > 0$ $G(x, t) = \displaystyle\int_{-\infty}^{+\infty}\frac{dk}{2\pi}\exp(-Dtk^2 + ixk)$
(v) Writing $Dtk^2 - ixk$ as $Dt(k - i\alpha)^2 + Dt\alpha^2$, evaluate $G(x, t)$ for $t > 0$, assuming the integral $\displaystyle\int_{-\infty}^{\infty} dk' e^{-\lambda k'^2} = \sqrt{\frac{\pi}{\lambda}}$.
(vi) Check that $G(x, t)$ gives a sensible temperature distribution for a single pulse of heat at $x' = t' = 0$.