P201 Workshop 10, Week 10

Please follow the instructions of your supervisor regarding timing of these problems.

Physics

1. Quantum computer **qubits** are vectors in two dimensions that represent, e.g., the spin of an electron or the internal state of a two-level atom. We represent the spin up state '$|\uparrow\rangle$' by the unit base vector $(1,0)$ and the the spin down state '$|\downarrow\rangle$' by the unit base vector $(0,1)$. A **one-qubit quantum gate** is a two-by-two matrix operating on spin (up and down) states or arbitrary sums (linear combinations) of spin up and spin down states.

   (i) Show that the **Hadamard-Gate**

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

   shifts the spin-up and spin-down states into the linear combinations (*superpositions*)

$$H|\uparrow\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle), \quad H|\downarrow\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle - |\downarrow\rangle).$$

   (ii) The **NOT-Gate** A performs the operation $A|\uparrow\rangle = |\downarrow\rangle$ and $A|\downarrow\rangle = |\uparrow\rangle$, i.e., it flips the spin. Find the explicit form of that two-by-two matrix A.

Math Practise

2. Let $A = \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}$, $\boldsymbol{x}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, $\boldsymbol{x}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

   (i) Calculate $A\boldsymbol{x}_1$ and $A\boldsymbol{x}_2$.

   (ii) Check that $2A\boldsymbol{x}_1 + A\boldsymbol{x}_2 = A(2\boldsymbol{x}_1 + \boldsymbol{x}_2)$.

3. Define the three Pauli-Spin-Matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

   (i) Calculate $\sigma_i \boldsymbol{e}_1$ and $\sigma_i \boldsymbol{e}_2$ for $i = x, y, z$. Here, $\boldsymbol{e}_1 = (1,0)$ and $\boldsymbol{e}_2 = (0,1)$.

   (ii) Calculate $\sigma_x \sigma_z \boldsymbol{x}$ for $\boldsymbol{x} = (1,1)$. Hint: First calculate $\boldsymbol{y} = \sigma_z \boldsymbol{x}$.

4. Calculate the determinants of $A = \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & c \\ c & 0 \end{pmatrix}$, $C = \begin{pmatrix} z & 1 \\ 1 & z^* \end{pmatrix}$.

Math Problems

5. (i) Show that the linear mappings described by

$$P_x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_y = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

   applied to a vector $\boldsymbol{x} = (x, y)$, yields the projection of that vector onto one of the coordinate axes (which?).
   Hint: Multiply P_x with $\boldsymbol{x}$ and P_y with $\boldsymbol{x}$. Make a sketch.
   (ii) Calculate $P_x(P_y\boldsymbol{x})$ and $P_y(P_x\boldsymbol{x})$.

6. Consider the matrix $S(\theta)$ that represent the reflection of vectors about a fixed axis,

$$S(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

   for the special case $\theta = \pi/2$. Show that $S(\theta = \pi/2) = \sigma_x$ is a reflection along the axis $x = y$. Hint: apply σ_x to the vectors $(1,0)$, $(0,1)$, and $(1,1)$.

7. We consider the rotation

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

   (i) Calculate $R(\theta)\boldsymbol{e}_1$ and $R(\theta)\boldsymbol{e}_2$ where $\boldsymbol{e}_1 = (1,0)$ and $\boldsymbol{e}_2 = (0,1)$. Make a sketch and show that $R(\theta)$ performs an anti-clockwise rotation in the x-y-plane.

   (ii) Consider an arbitrary vector $\boldsymbol{x} = (x, y)$. For which values of θ are the vectors $R(\theta)\boldsymbol{x}$ and $\boldsymbol{x}$ collinear?

   (iii) Calculate (by differentiating the matrix elements) $X = \frac{d}{d\theta}R(\theta)\Big|_{\theta=0}$ and use the result to calculate the first two terms in the Taylor expansion of $R(\theta)$ around $\theta = 0$. Show that $R(\theta) = I + X\theta + O(\theta^2)$, where I is the unit matrix.

   (iv) Apply the matrix $I + X\theta$ to $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$, make a sketch, and compare with the result for the full rotation from part 1.