P201 Workshop 11, Week 11

Please follow the instructions of your supervisor regarding timing of these problems.

Math Practice

1. For which of the following matrices does the inverse exist? Calculate the inverse if it exists.

$$\begin{aligned} A &= \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & c \\ c & 0 \end{pmatrix}, \quad C = \begin{pmatrix} z & 1 \\ 1 & z^* \end{pmatrix}, \\ P_x &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_y = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \\ \sigma_x &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \\ R(\theta) &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \end{aligned}$$

2. Calculate the following products and commutators (recall that the commutator of two matrices X and Y is $[X, Y] = XY - YX$)

(i) AB, BA, $[A, B]$;
(ii) $\sigma_x\sigma_y$, $\sigma_y\sigma_x$, $[\sigma_x, \sigma_y]$.

3. Show that rotations $R(\theta)$ in two dimensions commute, i.e., $R(\theta_1)R(\theta_2) = R(\theta_2)R(\theta_1)$.

(i) Prove this by a geometric argument (sketch!)

(ii) Prove this by calculating $R(\theta_1)R(\theta_2)$ for arbitrary θ_1 and θ_2 and using the addition theorem

$$\begin{aligned} \sin\alpha\sin\beta &= \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right], \\ \cos\alpha\cos\beta &= \frac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right], \\ \sin\alpha\cos\beta &= \frac{1}{2}\left[\sin(\alpha-\beta) + \sin(\alpha+\beta)\right]. \end{aligned}$$

4. Consider the matrix $S(\theta)$ that represent the reflection of vectors about an axis making an angle $\theta/2$ with the x axis,

$$S(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}.$$

Evaluate $S(\theta_1)S(\theta_2)$. Simplify and interpret your answer. Can you understand this result geometrically?

5. Evaluate $S(\pi/2)R(\theta)S(\pi/2)$. Simplify and interpret your answer. Can you understand this result geometrically?

Math Problems

6. The exponential function of a matrix A is defined as $e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}$. Consider the matrix

$$A = \lambda\sigma_z = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix},$$

use the definition $A^0 = I$ (unit matrix), calculate A^n for $n = 1, 2, 3, 4, \ldots$, and from this calculate e^A.

7. Calculate the eigenvalues and eigenvectors of the matrices listed in Problem 1, as well as $S(\theta)$. In this last case give a graphical interpretation of your answer.