P201 Workshop 3, Week 3

Please follow the instructions of your supervisor regarding timing of these problems.

Math Practise (no calculators allowed)

1. Calculate the inverse $1/z$ and the polar form $z = re^{i\phi}$ of the following:
(i) $z = i$; (ii) $z = -i$; (iii) $z = -1 + i$.

2. Calculate the following in Cartesian form, $z = x + iy$.
(i) $z = e^{\pi i}$; (ii) $z = 2e^{\pi i}$; (iii) $z = e^{(2n+1/2)\pi i}, n = 0, 1, 2, ...$

3. (i) Write the definition of $\sinh(x)$, $\cosh(x)$, $\tanh(x)$, and $\coth(x)$.

   (ii) Calculate the values of the functions in i) for $x = 0$. Sketch the functions in i).

   (iii) Calculate the derivative $\tanh'(x)$.

   *(iv) find an approximation for $\ln[\sinh(x)]$ for very large $x \gg 1$.

4. (i) For real x, simplify $\sinh(ix)$, $\cosh(ix)$, and $\tan(ix)$.

5. (i) Sketch the inverse hyperbolic cosine, $\cosh^{-1}(x)$.

6. Classify the following (linear or nonlinear, homogeneous or inhomogeneous):
(i) $y'' + y = x^2$; (ii) $y'' + y^2 = x$; (iii) $y'' + |y| = 0$; (iv) $y'' + xy = 1$.

Physics

7. Consider the weakly damped, harmonic oscillator as described by

$$\ddot{x}(t) + \gamma\dot{x}(t) + \omega^2 x(t) = 0, \quad \omega \gg \gamma > 0.$$

   Assume $x(t) = e^{izt}$ and insert this into the differential equation. Solve the resulting quadratic equation in order to obtain two values $z = z_1$ and $z = z_2$. Discuss the corresponding solutions $e^{iz_1 t}$ and $e^{iz_2 t}$.

8. The quantum Hall effect occurs in a two–dimensional sheet (x–y–plane) of electrons in a strong magnetic field B perpendicular to the plane. Non–interacting electrons are described by complex coordinates $z = x + iy$ in the x–y–plane, with wave functions

$$\Psi_m(z) = \frac{1}{\sqrt{2^{m+1}\pi m!}} \left(\frac{z}{l_B}\right)^m e^{-\left|\frac{z}{2l_B}\right|^2},$$

   where $l_B = (c/|eB|)^{1/2}$ is the typical lengthscale of this systems which is called 'magnetic length'. Sketch the probability $|\Psi_m(z)|^2$ as a function of $|z|$ for different quantum numbers m. Large m corresponds to large angular momentum: argue why this is consistent with your plots.

9. A particle of mass m moves on a line (x–axis). The only force acting on the particle is a friction $-\gamma\dot{x}(t)$ ($x(t)$: position of the particle at time t.)

   (i) Write down Newton's law for this problem.

   (ii) Solve the resulting differential equation. Hint: solve for the velocity first. Assume that a time $t = 0$, the position is $x(t = 0) = x_0$ and the velocity is $v(t = 0) = v_0$.

Math Problems

10. (i) Show that $y''(x) + p(x)y'(x) = f(x)$ can be transformed into a first order equation. Derive that equation and solve it for $f(x) \equiv 0$.

    (ii) Solve $y'(x) = e^{2x}, y(0) = 1$.

11. (i) Consider a star of mass $y(t)$ that varies as a function of time according to $y'(t) + r(t)y(t) = s(t)$, where $r(t)$ is the time–dependent rate of increase (or decrease) of the mass and $s(t)$ is a time–dependent mass source. Show that the solution at time t for the initial condition $y(0) = 0$ can be written using the integrating factor $g(t)$,

$$y(t) = \frac{1}{g(t)} \int_0^t dt' s(t') g(t'), \quad g(t) = \exp\left(\int_0^t dy r(y)\right).$$

    Hint: write $1/g(t)$ as $\exp(-\ldots)$, and verify the expression for $y(t)$ by calculating $y'(t)$. Discuss limiting cases of this general formula, such as $s(t) = 0$, $r(t) = r = const$.

    (ii) Solve $y'(x) - \tan(x)y(x) = \cos(x), y(0) = 0$. Hint: you can use part i).

No assigned reading for next week: Coursework due