P201 Workshop 9, Week 9

Please follow the instructions of your supervisor regarding timing of these problems.

Physics

1. Show that from Planck's law

$$u(\nu, T) = \frac{8\pi\nu^2}{c^3} \frac{h\nu}{\exp\left(\frac{h\nu}{k_B T}\right) - 1},$$

the Wien law

$$u(\nu, T) = \frac{4\nu^3}{c^3} b \exp\left(-\frac{a\nu}{T}\right), \quad a, b = \text{constants}$$

and the Rayleigh–Jeans law

$$u(\nu, T) = \rho(\nu)\bar{E}(\nu) = \frac{8\pi\nu^2}{c^3} k_B T,$$

follow as limiting cases.

2. (i) Calculate the first and the second derivatives of the Lennard–Jones Potential

$$V(r) = V_0 \left[\left(\frac{a}{r}\right)^{12} - 2\left(\frac{a}{r}\right)^6 \right], \quad a > 0, \quad r > 0.$$

(ii) Calculate the position r_0 where the potential has its minimum, and Taylor–expand it $V(r)$ around this minimum. Sketch $V(r)$ and its 'harmonic (parabolic) approximation'.

*(iii) Determine the frequency for small oscillations of a mass m around the minimum r_0.

Math Practise

3. Find the first three terms in the Taylor expansion for small $|x|$ around $x = 0$ of the following functions:

(i) $f(x) = \sqrt{1+x}$; (ii) $f(x) = \arctan(x)$; (iii) $f(x) = 1/\sqrt{1+x}$;

(iv) $f(x) = 1/\sqrt{1-x}$; (v) $f(x) = \sin(x)/x$.

4. Find the Taylor expansion (all terms) for small $|x|$ around $x = 0$ of the following functions:

(i) $f(x) = 1/(1+x)$.

(ii) $f(x) = \cosh(x)$.

5. (i) Use the Taylor expansion of $f(x) = \sqrt{1+x}$ to approximately calculate $\sqrt{10}$. Hint: Write $10 = 9 + 1 = 9(1 + 1/9)$.

(ii) Approximately calculate a) $\sqrt{143}$ and b) $\sqrt{100 - a}$, $|a| < 10$.

Math Problems

6. Taylor Expansions:

(i) Use $\ln \frac{1+x}{1-x} = \ln(1+x) - \ln(1-x)$ and show that for $|x| < 1$,

$$\ln \frac{1+x}{1-x} = 2 \sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k+1}.$$

Use this to find an approximate value of $\ln 2$.

(ii) Find the Taylor expansion (all terms) for small $|x|$ around $x = 0$ of

$$f(x) = \frac{1}{1+x^n}, \quad n \geq 1 \text{ integer and arbitrary.}$$

7. Expand the following functions around $x = 0$ (first two or three terms), sketch them in the vicinity of $x = 0$, and find $\lim_{x \to 0} f(x)$ (can be infinity in some cases):

(i) $f(x) = \sin(x)/x$. (ii) $f(x) = \sin^2(x)/x$. (iii) $f(x) = \cos(x)/x$.

(iv) $f(x) = cx/(1+x)$. (v) $f(x) = x/\sin(x)$.

(vi) $f(x) = 1/(e^{\beta x} - 1)$, $\beta > 0$ (Bose distribution) .

No assigned reading for next week: Coursework due