Solutions to Sheet 11

Notations: Det for det, . for matrix/matrix product, Inverse[] for power -1.

1.

(i)

$$A = \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix};\ B = \begin{pmatrix} 1 & c \\ c & 0 \end{pmatrix};\ \mathrm{Cc} = \begin{pmatrix} z & 1 \\ 1 & z^* \end{pmatrix};\ P_x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix};\ P_y = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix};$$
$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};\ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix};\ R(\theta\_) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix};$$

Det(*A*)

-5

Inverse[*A*]

$$\begin{pmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{3}{5} & -\frac{2}{5} \end{pmatrix}$$

Det[*B*]

$-c^2$

Inverse[*B*]

$$\begin{pmatrix} 0 & \frac{1}{c} \\ \frac{1}{c} & -\frac{1}{c^2} \end{pmatrix}$$

Det[Cc]

$z z^* - 1$

Inverse[Cc]

$$\begin{pmatrix} \frac{z^*}{z z^* - 1} & -\frac{1}{z z^* - 1} \\ -\frac{1}{z z^* - 1} & \frac{z}{z z^* - 1} \end{pmatrix}$$

Det[P_x]

0

$\mathbf{Det}\left[P_y\right]$

0

$\mathbf{Det}[\sigma_x]$

-1

$\mathbf{Inverse}[\sigma_x]$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$\mathbf{Det}\left[\sigma_y\right]$

-1

$\mathbf{Inverse}\left[\sigma_y\right]$

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$\mathbf{Simplify}[\mathbf{Det}[R(\theta)]]$

1

$\mathbf{Simplify}[\mathbf{Inverse}[R(\theta)]]$

$$\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$$

So R(- θ) !

2. Math practice

(i)

A.B

$$\begin{pmatrix} c+2 & 2c \\ 3-c & 3c \end{pmatrix}$$

B.A

$$\begin{pmatrix} 3c+2 & 1-c \\ 2c & c \end{pmatrix}$$

$$\boldsymbol{A.B - B.A}$$

$$\begin{pmatrix} -2\,c & 3\,c-1 \\ 3-3\,c & 2\,c \end{pmatrix}$$

(ii)

$$\boldsymbol{\sigma_x.\sigma_y}$$

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$\boldsymbol{\sigma_y.\sigma_x}$$

$$\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

$$\boldsymbol{\sigma_x.\sigma_y - \sigma_y.\sigma_x}$$

$$\begin{pmatrix} 2\,i & 0 \\ 0 & -2\,i \end{pmatrix}$$

3.

(i) Rather trivial, the second rotation rotates the initial unit vectors by another angle θ_1. So the result is a rotation over $\theta = \theta_1 + \theta_2$. In this addition order doesn't matter.

(ii)

$$\textbf{Simplify}[\boldsymbol{R(\theta_1).R(\theta_2)}]$$

$$\begin{pmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{pmatrix}$$

4. Reflections

$$\boldsymbol{S(\theta\_) = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix};}$$

$$\textbf{Simplify}[\boldsymbol{S(\theta_1).S(\theta_2)}]$$

$$\begin{pmatrix} \cos(\theta_1-\theta_2) & -\sin(\theta_1-\theta_2) \\ \sin(\theta_1-\theta_2) & \cos(\theta_1-\theta_2) \end{pmatrix}$$

This is a rotation over an angle $\theta_1 - \theta_2$.

5.

$S(\pi/2).R(\theta).S(\pi/2)$

$$\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$$

So you'll have to show graphically that this is a rotation over - θ!

6.

$A^n = \lambda^n I$ if n is even, $\lambda^n \sigma_z$ if n is odd. Thus $\exp(\lambda\,\sigma_z) = \begin{pmatrix} \sum_{n=0}^{\infty} \lambda^n/n! & 0 \\ 0 & \sum_{n=0}^{\infty} (-\lambda)^n/n! \end{pmatrix} = \begin{pmatrix} \exp(\lambda) & 0 \\ 0 & \exp(-\lambda) \end{pmatrix}$.

7. Rotations

Eigenvalues[A]

$$\left\{\frac{1}{2}\left(1-\sqrt{21}\right), \frac{1}{2}\left(1+\sqrt{21}\right)\right\}$$

Print[First[Eigenvectors[A]], ",", Last[Eigenvectors[A]]]

$$\left\{\frac{1}{6}\left(3-\sqrt{21}\right), 1\right\},\left\{\frac{1}{6}\left(3+\sqrt{21}\right), 1\right\}$$

Eigenvalues[B]

$$\left\{\frac{1}{2}\left(1-\sqrt{4c^2+1}\right), \frac{1}{2}\left(\sqrt{4c^2+1}+1\right)\right\}$$

Print[First[Eigenvectors[B]], ",", Last[Eigenvectors[B]]]

$$\left\{-\frac{\sqrt{4c^2+1}-1}{2c}, 1\right\},\left\{-\frac{-\sqrt{4c^2+1}-1}{2c}, 1\right\}$$

Eigenvalues[Cc]

$$\left\{\frac{1}{2}\left(z+z^*-\sqrt{z^2-2z^*z+(z^*)^2+4}\right), \frac{1}{2}\left(z+z^*+\sqrt{z^2-2z^*z+(z^*)^2+4}\right)\right\}$$

Can be rewritten as $\mathrm{Re}(z) \pm \sqrt{\mathrm{Im}(z)^2+1}$

Print[First[Eigenvectors[Cc]], ",", Last[Eigenvectors[Cc]]]

$\left\{\frac{1}{2}\left(z - z^* - \sqrt{z^2 - 2\,z^*\,z + (z^*)^2 + 4}\right), 1\right\}, \left\{\frac{1}{2}\left(z - z^* + \sqrt{z^2 - 2\,z^*\,z + (z^*)^2 + 4}\right), 1\right\}$

Eigenvalues[P_x]

{0, 1}

Print[First[Eigenvectors[P_x]], ",", Last[Eigenvectors[P_x]]]

{0, 1},{1, 0}

Eigenvalues[P_y]

{0, 1}

Print[First[Eigenvectors[P_y]], ",", Last[Eigenvectors[P_y]]]

{1, 0},{0, 1}

Eigenvalues[σ_x]

{−1, 1}

Print[First[Eigenvectors[σ_x]], ",", Last[Eigenvectors[σ_x]]]

{−1, 1},{1, 1}

Eigenvalues[σ_y]

{−1, 1}

Print[First[Eigenvectors[σ_y]], ",", Last[Eigenvectors[σ_y]]]

{i, 1},{−i, 1}

Eigenvalues[$R(\theta)$]

$\{\cos(\theta) - i\sin(\theta), \cos(\theta) + i\sin(\theta)\}$

Print[First[Eigenvectors[$R(\theta)$]], ",", Last[Eigenvectors[$R(\theta)$]]]

{−i, 1},{i, 1}

Eigenvalues[$S(\theta)$]

$\{-1, 1\}$

Print[First[Eigenvectors[$S(\theta)$]], ",", Last[Eigenvectors[$S(\theta)$]]]

$\{(\cos(\theta) - 1)\csc(\theta), 1\}$,$\{(\cos(\theta) + 1)\csc(\theta), 1\}$

Can be rewritten as $\sin(\theta)\,(\cos(\theta) \pm 1, \sin(\theta))$. This vectors can be simplified to $2\cos(\theta/2)\,(\cos(\theta/2), \sin(\theta/2))$ and $2\sin(\theta/2)\,(-\sin(\theta/2), \cos(\theta/2))$. The most elegant form (also the unit-length form) is thus ($\cos(\theta/2)$,$\sin(\theta/2)$) and ($-\sin(\theta/2)$,$\cos(\theta/2)$). Check which eigenvalue these correspond to....