Solutions to Sheet 5

1. LRC circuit, no external potential

(i) Rewrite the first two relations as $I(t) = -\dot{Q}(t) = -C\,\dot{U}(t)$, and find $U(t) = -L\,C\,.\ddot{U}(t) - R\,C\,\dot{U}(t) \Rightarrow \ddot{U}(t) + \frac{R}{L}\,\dot{U}(t) + \frac{1}{L\,C}\,U(t) = 0$.

(ii) This is a linear and homogeneous differential equation.

(iii) $R = 0$: $\ddot{U}(t) + \frac{1}{L\,C}\,U(t) = 0$. If $U(t) = e^{i\,z\,t}$, $z^2 = \frac{1}{L\,C} \Rightarrow z_{1,2} = \pm\,\frac{1}{\sqrt{\mathrm{LC}}}$. Thus $U_{1,2}(t) = \exp\left(\pm\,\frac{i}{\sqrt{\mathrm{LC}}}\,t\right)$.

(iv) Clearly, with $\omega = \frac{1}{\sqrt{\mathrm{LC}}}$, $U_{1,2} = \exp(\pm\,i\,\omega_0\,t) = \cos(\omega_0\,t) \pm i\sin(\omega_0\,t)$. Since the general solution is a linear superposition of these two, we can have any combination of sines and cosines $\Rightarrow$ $U(t) = A\cos(\omega_0\,t)\; + B\sin(\omega_0\,t)$.

(v) $U(t) = e^{i\,z\,t}$, $-z^2 + i\,\frac{R}{L}\,z + \frac{1}{L\,C} \Rightarrow z_{1,2} = +i\,\frac{R}{2\,L} \pm \frac{1}{2}\sqrt{-\frac{R^2}{L^2} + \frac{4}{L\,C}}$. If $R^2/L^2 < 4/(L\,C)$ (i.e., $R < 2\sqrt{L/C}$) the square root is real, and we have, with α and ω as on the sheet, $z_{1,2} = i\,\alpha \pm \omega$.

(vi)
$$\begin{aligned} U_{1,2} &= \exp(-\alpha\,t \pm i\,\omega\,t) \\ &= \exp(-\alpha\,t)\exp(\pm\,i\,\omega\,t) = \exp(-\alpha\,t)\,(\cos(\omega\,t) + i\sin(\omega\,t)) \end{aligned}$$

In the same way as in 4, the real solution is thus a superposition of exp(-α t)cos(ω t) and $\exp(-\alpha\,t)\sin(\omega\,t)$,

$U(t) = \exp(-\alpha\,t)\,(A\cos(\omega\,t) + B\sin(\omega\,t))$.

(vii) If $R = 0$, $\alpha = 0$ and $\omega = \omega_0$, so we trivially find the requested result.

2. LRC circuit, sinusoidal potential

(i) Trivial differentiation, plus substitution of $I(t) = -\dot{Q}(t)$.

(ii) The equation (2) is linear and inhomogeneous.

(iii) $\left(-\Omega^2\,L + i\,\Omega\,R + 1/C\right) I_0\,e^{i\,\Omega\,t} = \Omega\,V_0\,e^{i\,\Omega\,t}$, or

$$\begin{aligned} I_0 &= \frac{\Omega\,V_0}{\left(-\Omega^2\,L + i\,\Omega\,R + 1/C\right)} = \\ &\frac{V_0(-\Omega\mathrm{L} + 1/(\Omega\,C) - i\;R)}{(-\Omega\mathrm{L} + 1/(\Omega\,C) + i\;R)\,(-\Omega\mathrm{L} + 1/(\Omega\,C) - i\;R)} = \frac{V_0(-\Omega\mathrm{L} + 1/(\Omega\,C) - i\;R)}{(-\Omega\mathrm{L} + 1/(\Omega\,C))^2 + R^2}\,. \end{aligned}$$

(iv) $I_0 = \frac{V_0(-\Omega\mathrm{L}+1/(\Omega\,C)-i\;R)}{(-\Omega\mathrm{L}+1/(\Omega\,C))^2+R^2}$. $|\,I_0\,| = \frac{V_0}{\sqrt{(-\Omega\mathrm{L}+1/(\Omega\,C))^2+R^2}}$, $\phi = \arg(I_0) = \mathrm{atan}\left(\frac{R}{-\Omega\mathrm{L}+1/(\Omega\,C)}\right)$

(v) a) taking the complex conjugate of the auxiliary equation, we must show that $\left(\dot{I}(t)\right)^* = \frac{d}{\mathrm{dt}}\,(I(t)^*)$. This is straightforward one we realise that t is real, so that

$$\left(\dot{I}(t)\right)^* = \left(\lim_{h\to 0} \frac{I(t+h) - I(t)}{h}\right)^* = \lim_{h\to 0} \frac{I(t+h)^* - I(t)^*}{h} = \frac{d}{\mathrm{dt}} (I(t)^*).$$

b) Adding the equation and its complex conjugate we get the desired answer.

(vi)Trivially true! If $R = 0$, ϕ=0, and the phase difference is 90°.

3.

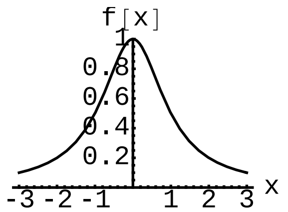
(i) take $y(x) = \exp(i\, z\, x)$: $-z^2 = -12$, $z = \pm\sqrt{12}$: $y(x) = A\cos\left(\sqrt{12}\ x\right) + B\sin\left(\sqrt{12}\ x\right)$.

(ii) take $y(x) = \exp(i\, z\, x)$: $-z^2 + i\, z + 12 = 0$, $z = -\frac{1}{2}\, i \pm \frac{1}{2}\sqrt{-1+48}$:

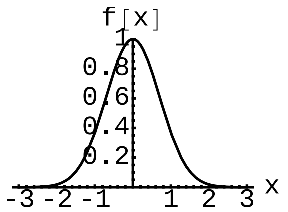
$y(x) = \exp\left(\frac{1}{2}\, x\right)\left(A\cos\left(\sqrt{47/4}\ x\right) + B\sin\left(\sqrt{47/4}\ x\right)\right)$.

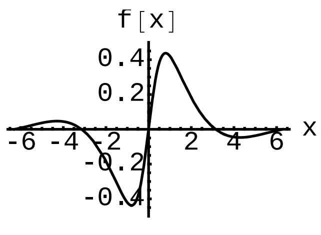
4.

(i)



(ii)





5.

(i) $e^{i\,3\,\pi/2}$

(ii) $2\, e^{-i\,\pi/3}$

(iii) $\left(2\, e^{i\,\pi/3}\right)^{1/4} = 2^{1/4}\, e^{i\,\pi/12}$

(iv) One solution $\left(2\, e^{-2\,i\,\pi/3}\right)^{1/4} = 2^{1/4}\, e^{-\iota\,\pi/6}$. Other solutions differ by phase by $2\,\pi/4 = \pi/2$,

$z = 2^{1/4}\, e^{-i\,2\,\pi/3}$, $2^{1/4}\, e^{-i\,\pi/6}$, $2^{1/4}\, e^{i\,\pi/3}$, $2^{1/4}\, e^{i\,5\,\pi/6}$.

6.

(i) Subsitute: $-9\,C\,e^{i\,3\,x} + 4\,C\,e^{i\,3\,x} = 2\,e^{i\,3\,x}$: $C = -2/5$.

(ii) General solution of the homegeneous equation $y(x) = A\ e^{i\,2\,x} + B\,e^{-i\,2\,x}$. Solution to inhomogeneous problem: $y(x) = A\ e^{i\,2\,x} + B\,e^{-i\,2\,x} - \frac{2}{5}\,e^{i\,3\,x}$. $y(0) = 0$, $y'\,(0) = 0$:

$A + B - \frac{2}{5} = 1$, $i\left(2\,A\ - 2\,B - \frac{6}{5}\right) = 0$.

Solution: $A = 1$, $B = \frac{2}{5}$.

$y(x) = e^{i\,2\,x} + \frac{2}{5}\,e^{-i\,2\,x} - \frac{2}{5}\,e^{i\,3\,x}$.