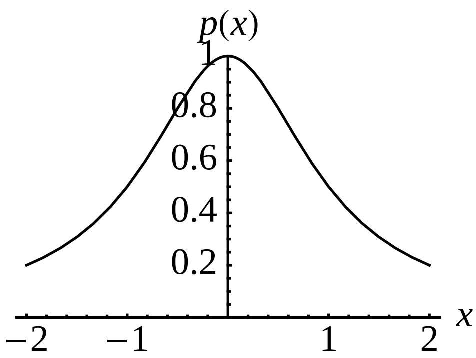
Solutions to Sheet 2

1. Quantum wire

$p(x) = \frac{c}{x+i}\,\frac{c}{x-i} = \frac{c^2}{x^2+1}$. This has a maximum at $\frac{d}{dx}\frac{1}{x^2+1} = -\frac{2x}{(x^2+1)^2} = 0$, i.e. x=0. This can be seen from a plot

Plot$\left[\frac{1}{x^2+1}, \{x, -2, 2\}\right.$, AxesLabel → {$x$, $p(x)$},
FormatType → TraditionalForm, PlotRange → {0, 1}]



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2. AC circuit

(i) We have $Y = \frac{1}{R} - \frac{i}{\omega L}$. Rewriting this in modulus and argument, $|Y| = \sqrt{\frac{1}{R^2} + \frac{1}{(\omega L)^2}} = \frac{1}{R\,\omega\,L}\sqrt{R^2 + (\omega L)^2}$, and $\arg(Y) = \arctan(R/(\omega L))$. Clearly, $|I_0| = \frac{V_0}{R\,\omega\,L}\sqrt{R^2 + (\omega L)^2}$.

(ii)

$$Z = R - \frac{i}{\omega C},\ I_0 = V_0/Z = \frac{V_0(R + i/(\omega C))}{R^2 + 1/(\omega C)^2} = \frac{V_0\left(R\,\omega^2\,C^2 + i\,\omega\,C\right)}{R^2\,\omega^2\,C^2 + 1},$$

$$V_C = I_0\,Z_C = V_0\,\frac{\left(R\,\omega^2\,C^2 + i\,\omega\,C\right)}{R^2\,\omega^2\,C^2 + 1}\left(-\frac{i}{\omega\,C}\right) = \frac{V_0(-i\,R\,\omega\,C + 1)}{R^2\,\omega^2\,C^2 + 1},$$

$$|V_C| = V_0\,\frac{\sqrt{R^2\,\omega^2\,C^2 + 1}}{R^2\,\omega^2\,C^2 + 1} = V_0\,\frac{1}{\sqrt{R^2\,\omega^2\,C^2 + 1}}$$

Since ω has the dimensions 1/time, $R\,C$ has the deimensions of time, and thus the typical time scale of such a circuit is $\tau = R\,C$.

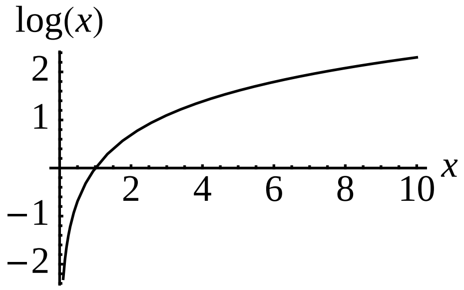
3. practice 1

(i) $z = x + i\,y$, $z^* = x - i\,y$, so $z + z^* = 2\,x$, $z - z^* = 2\,i\,y$. From $\mathrm{Re}(z) = x$, $\mathrm{Im}(z) = y$ we obtain $\mathrm{Re(z)} = \frac{1}{2}\,(z + z^*)$, $\mathrm{Im(z)} = \frac{1}{2\,i}\,(z - z^*)$.

(ii) Use $e^{i\,\theta} = \cos(\theta) + i\,\sin(\theta)$, and (i):
$\cos(\theta) = \mathrm{Re}\left(e^{i\,\theta}\right) = \frac{1}{2}\left(e^{i\,\theta} + e^{-i\,\theta}\right)$, $\sin(\theta) = \mathrm{Im}\left(e^{i\,\theta}\right) = \frac{1}{2\,i}\left(e^{i\,\theta} - e^{-i\,\theta}\right)$.

(iii)

Plot[Log[*x*], {*x*, 0.1, 10}, AxesLabel → {*x*, ln(*x*)}, FormatType → TraditionalForm]



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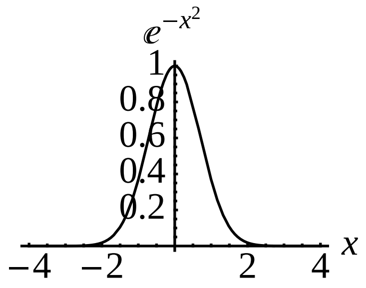
Plot[Exp[$-x^2$], {*x*, −4, 4}, AxesLabel → {*x*, exp($-x^2$)}, FormatType → TraditionalForm]



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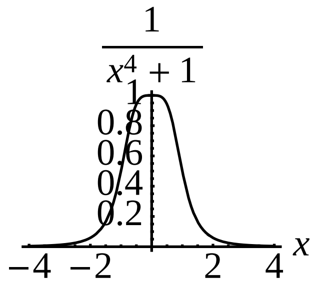
Plot$\left[\frac{1}{1+x^4}, \{x, -4, 4\}, \text{AxesLabel} \to \left\{x, \frac{1}{1+x^4}\right\}, \text{FormatType} \to \text{TraditionalForm}\right]$



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(iv) $\frac{1}{z} = \frac{z^*}{|z|^2}$, $\frac{1}{z^*} = \frac{z}{|z|^2} = \left(\frac{z^*}{|z|^2}\right)^*$.

4. polar representation

(i) $i = e^{i\pi/2}$, $\sqrt{i} = e^{i\pi/4}$

(ii) $(1+i) = \sqrt{2}\, e^{i\pi/4}$, $\sqrt{(1+i)} = 2^{1/4}\, e^{i\pi/8}$

(iii) $(1+i)^2 = 2\, e^{i\pi/2}$ (= $2\,i$) (used result (ii)above).

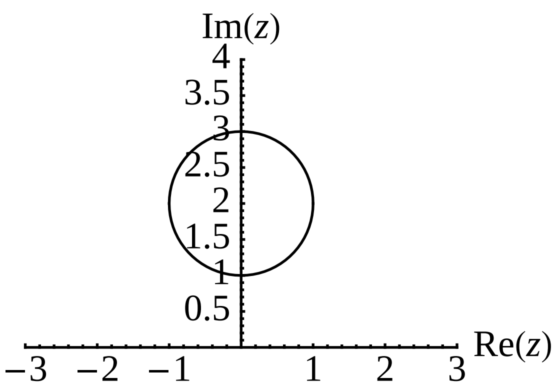
(iv) $(1+i)^4 = 4\, e^{i\pi}$ (= -4)

5. Inequalities

(i) for the same reason as for vectors: a complex number has two parts, and inequality really refers to a single number.

(ii) This is the interior of a circle: $\sqrt{x^2 + (y-2)^2} < 1$

ContourPlot$\left[x^2 + (y-2)^2 == 1, \{x, -1, 1\}, \text{AxesLabel} -> \{\text{Re}(z), \text{Im}(z)\},\right.$
$\left.\text{PlotRange} \to \{\{-3, 3\}, \{0, 4\}\}, \text{FormatType} \to \text{TraditionalForm}\right]$



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6. Knowledge questions

(i) $\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

(ii) $\mathrm{Re}\left(e^{-i\,x}\right) = \cos(-x) = \cos(x)$, $\mathrm{Im}\left(e^{-i\,x}\right) = \sin(-x) = -\sin(x)$

(iii) i, -1, 1

(iv) $\mathrm{Re}\left(e^{-\gamma\,t+i\omega\,t}\right) = e^{-\gamma t} \cos(\omega\,t)$. Limiting value zero for $\gamma>0$, undefined for $\gamma<0$.

7. Math Problems

(i) z=5-i

```
Show[Graphics[{Arrow[{0, 0}, {2, 1}], Arrow[{0, 0}, {3, −2}], Hue[0], Arrow[{0, 0}, {5, −1}]}],
  Axes → True]
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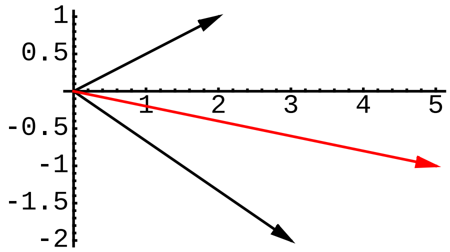

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(ii) $|z_1| = \sqrt{5}$, $|z_2| = \sqrt{13}$. These are the length of the vectors.
(iii) draw a circle of the sum diameter, and show it is larger.

(iv) $z = \sqrt{x^2 + y^2}\ \exp(\arg(y/x))$

(v) $z = \sqrt{2}\ e^{i\,\pi}$
(vi)

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Show[Graphics[{Disk[{0, 0}, 2], RGBColor[1, 1, 1], Disk[{0, 0}, 1]}],
  Frame → True, AspectRatio → 1]
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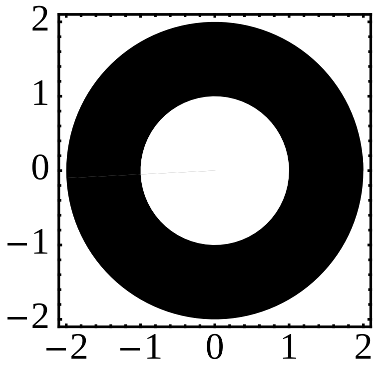

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7. de Moivre's theorem

(i) $z = e^{i\theta} = (x + i\,y)$. $z^3 = x^3 - 3\,x\,y^2 + i\left(3\,x^2\,y - y^3\right)$.

Conclude $\cos(3\,\theta) = \cos^3(\theta) - 3\cos(\theta)\sin^2(\theta) = 2\cos^3(\theta) - 3\cos(\theta)$,
$\sin(3\,\theta) = 3\cos^2(\theta)\sin(\theta) - \sin^3(\theta) = 3\sin(\theta) - 4\sin^4(\theta)$.

(ii) $z^4 = x^4 - 6\,x^2\,y^2 + y^4 + i\left(4\,x^3\,y - y^3\,x\right)$.

Conclude

$\cos(4\,\theta) = \cos^4(\theta) - 6\cos^2(\theta)\sin^2(\theta) + \sin^4(\theta) = 8\cos^4(\theta) - 8\cos^2(\theta) + 1$,
$\sin(4\,\theta) = 4\cos^3(\theta)\sin(\theta) - 4\sin^3(\theta)\cos(\theta)$.