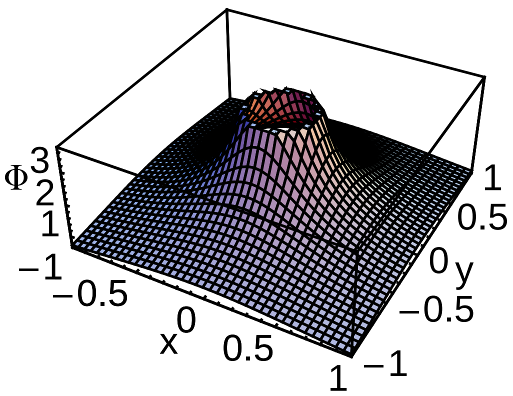
Solutions to Sheet 7

1. PVT surface

(i)

```
Plot3D[1/Sqrt[x^2 + y^2], {x, -1, 1}, {y, -1, 1},
  PlotPoints → 41, ClippingStyle → None, AxesLabel → {x, y, Φ}]
```



- SurfaceGraphics -

(ii) $d\,\Phi = \frac{\partial}{\partial x}\Phi\,d\,x + \frac{\partial}{\partial y}\Phi \mathrm{d}\,y = \frac{-1}{\left(x^2+y^2\right)^{3/2}}\,(x\,d\,x + y\,d\,y).$

$-e\,\frac{d\,\Phi}{d\,t} = \frac{e}{\left(x^2+y^2\right)^{3/2}}\left(x\,\frac{d\,x}{d\,t} + y\,\frac{d\,y}{d\,t}\right)$

(iii) $\frac{e}{r^3}\,(r\cos(\omega\,t)\,(-\omega)\,r\sin(\omega\,t) + r\sin(\omega\ t)\,\omega\,r\cos(\omega\,t)) = 0.$

(iv) $\frac{e}{(a+v\,t)^3}\,((a+v\,t)\,v + (a+v\,t)\,v) = \frac{e\,v}{(a+v\,t)^2}.$

2.

(i) $z = x^2 + y^2,\ \partial_x\cosh(z) = \sinh(z)\,\partial_x z = 2\,x\sinh\left(x^2+y^2\right)$

(ii) $a = \sin(x\,y),\ \ b = x\,y,\ \partial_y\sinh(a) = \frac{d}{d\,a}\sinh(a)\,\frac{d}{d\,b}\sin(b)\,\partial_y b = \cosh(\sin(x\,y))\cos(x\,y)\,x.$

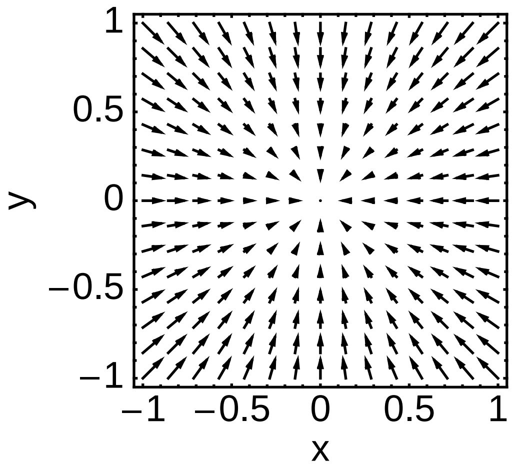
3.

(i) $\nabla\ f(x,\,y)$ =(- 2x,- 2y)

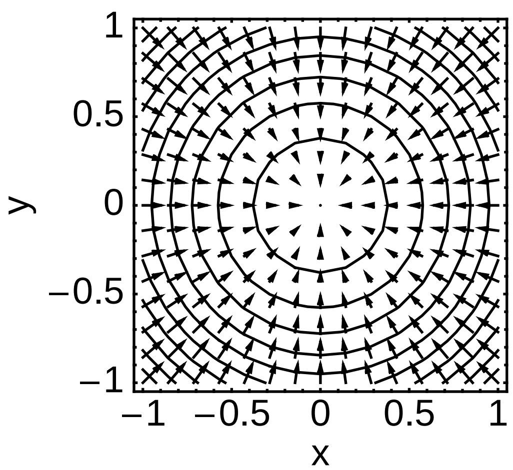
(ii)

```
<< "VectorFieldPlots`"
```

```
p1 = (Needs["VectorFieldPlots`"];
   VectorFieldPlots`VectorFieldPlot[{-2 x, -2 y},
    {x, -1, 1}, {y, -1, 1}, Frame → True, FrameLabel → {x, y}]);
```



Show$\left[\mathbf{p1},\ \mathbf{ContourPlot}\left[-\left(x^2+y^2\right),\ \{x,-1,1\},\ \{y,-1,1\},\ \mathbf{Axes}\to\mathbf{True},\right.\right.$
$\left.\left.\mathbf{AxesLabel}\to\{x,y\},\ \mathbf{ContourShading}\to\mathbf{False},\ \mathbf{DisplayFunction}\to\mathbf{Identity}\right]\right]$



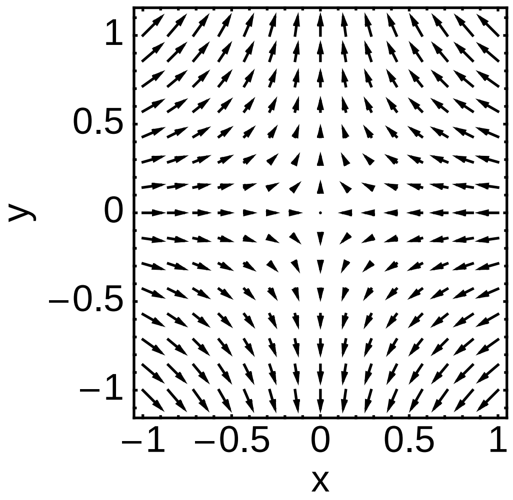
- Graphics -

4.

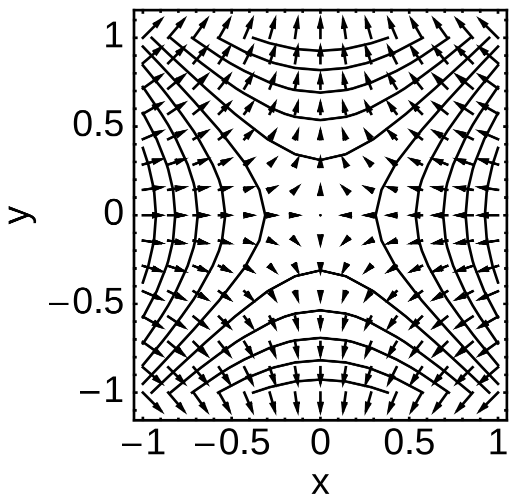
(i) $\nabla\ f(x,y)$ =(- 2x,2y)

(ii)

```
p1 = (Needs["VectorFieldPlots`"];
   VectorFieldPlots`VectorFieldPlot[{-2 x, 2 y}, {x, -1, 1},
    {y, -1, 1}, Frame → True, FrameLabel → {x, y}]);
```




```
Show[p1, ContourPlot[-x^2 + y^2, {x, -1, 1}, {y, -1, 1}, Axes → True,
   AxesLabel → {x, y}, ContourShading → False, DisplayFunction → Identity]]
```



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5.

(i) a) $\nabla f = (0, 1)$: no stationary points

b) $\nabla f = (1, 0)$: no stationary points

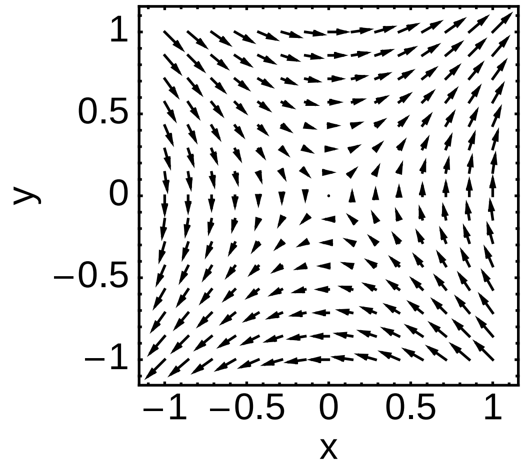
c) $\nabla f = -4\left(x^3, y^3\right)$: stationary points $(x, y) = 0$

d) $\nabla f = (y, x)$: stationary points (x,y)=0

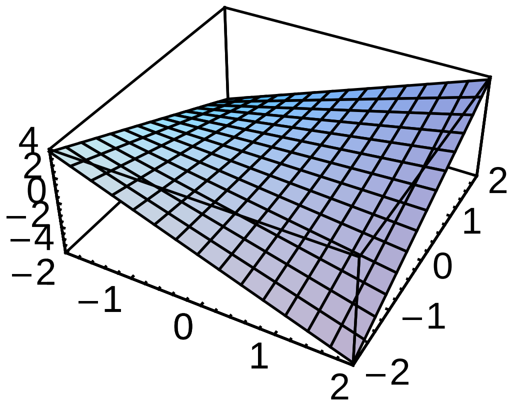
e) $\nabla f = 4\left(x^3, -y^3\right)$:stationary points $(x, y) = 0$

(ii)

```
p1 = (Needs["VectorFieldPlots`"]; VectorFieldPlots`VectorFieldPlot[
    {y, x}, {x, -1, 1}, {y, -1, 1}, Frame → True, FrameLabel → {x, y}]);
```



Plot3D[*x y*, {*x*, −2, 2}, {*y*, −2, 2}]



- SurfaceGraphics -

Everything suggests that this is a saddle, with axis rotated by 45^o.

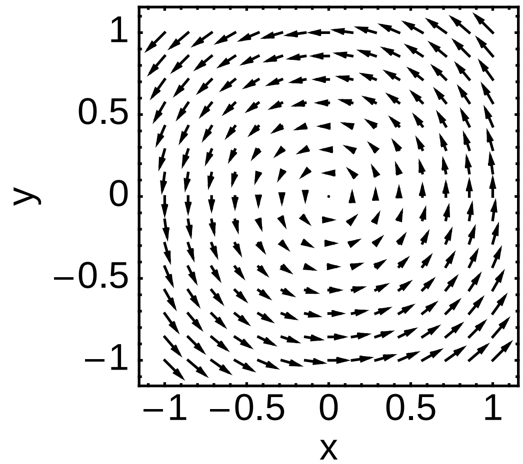
6.

A plane is defined by an equation $z = z_0 + a\,x + b\,y$. The fiven equation is of that form.
∇f=(0,2), so $f_1(x, y) = 4 + 2\,y$. This is the collection of all lines $z = 4 + 2\,y$ for all x, and makes a plane

7.

(i)

```
(Needs["VectorFieldPlots`"]; VectorFieldPlots`VectorFieldPlot[
   {-y, x}, {x, -1, 1}, {y, -1, 1}, Frame → True, FrameLabel → {x, y}]);
```



(ii) Solutions should follow the arrows → circles!

(iii)

DSolve[$y'[x] == -x/y[x]$, $y[x]$, x]

$$\left\{\left\{y(x) \to -\sqrt{2}\,\sqrt{c_1 - \frac{x^2}{2}}\right\}, \left\{y(x) \to \sqrt{2}\,\sqrt{c_1 - \frac{x^2}{2}}\right\}\right\}$$

$$\mathbf{Expand}\left[\left(\sqrt{2}\,\sqrt{c_1 - \frac{x^2}{2}}\right)^2 + x^2\right]$$

$2\,c_1$

So these are circles.

8.

(i) $y(x) = C_1\, e^{2\,i\,x} + C_2\, e^{-2\,i\,x}$

(ii) $-9\,C + 4\,C = 2 \Rightarrow C = \frac{-2}{5}$

(iii) General solution $y(x) = C_1\, e^{2\,i\,x} + C_2\, e^{-2\,i\,x} - \frac{2}{5}\, e^{3\,i\,x}$.

$y(0) = 1 \Rightarrow\ C_1 + C_2 - \frac{2}{5} = 1$

$y'\ (0) = 0 \Rightarrow 2\,i\,C_1 + 2\,i\,C_2 - \frac{6}{5}\,i = 0$

Solution: $C_1 \to 1,\ C_2 \to \frac{2}{5}$. Thus $y(x) = e^{2\,i\,x} + \frac{2}{5}\, e^{-2\,i\,x} - \frac{2}{5}\, e^{3\,i\,x}$