Solutions to Sheet 10

1. Quantum gates

(i)

$$H\,|\uparrow\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\cdot 1+1\cdot 0\\ 1\cdot 1-1\cdot 0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix} = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right)$$

$$H\,|\downarrow\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\cdot 0+1\cdot 1\\ 1\cdot 0-1\cdot 1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix} = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle - |\downarrow\rangle\right)$$

(ii) Write $A = \begin{pmatrix}a & b\\ c & d\end{pmatrix}$. Now we require that $A\begin{pmatrix}1\\ 0\end{pmatrix} = \begin{pmatrix}0\\ 1\end{pmatrix}$ and $A\begin{pmatrix}0\\ 1\end{pmatrix} = \begin{pmatrix}1\\ 0\end{pmatrix}$. Thus we find that $\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = \begin{pmatrix}a\cdot 1+b\cdot 0\\ c\cdot 1+d\cdot 0\end{pmatrix} = \begin{pmatrix}a\\ c\end{pmatrix} = \begin{pmatrix}0\\ 1\end{pmatrix}$ and $\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix} = \begin{pmatrix}a\cdot 0+b\cdot 1\\ c\cdot 0+d\cdot 1\end{pmatrix} = \begin{pmatrix}b\\ d\end{pmatrix} = \begin{pmatrix}1\\ 0\end{pmatrix}$. Thus $A = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$.

2. Math practice

(i)

$$A\,x_1 = \begin{pmatrix}2 & 1\\ 3 & -1\end{pmatrix}\begin{pmatrix}1\\ -2\end{pmatrix} = \begin{pmatrix}2\cdot 1+1\cdot(-2)\\ 3\cdot 1-1\cdot(-2)\end{pmatrix} = \begin{pmatrix}0\\ 5\end{pmatrix}$$

$$A\,x_2 = \begin{pmatrix}2 & 1\\ 3 & -1\end{pmatrix}\begin{pmatrix}1\\ 3\end{pmatrix} = \begin{pmatrix}2\cdot 1+1\cdot 3\\ 3\cdot 1-1\cdot 3\end{pmatrix} = \begin{pmatrix}5\\ 0\end{pmatrix}$$

(ii)

$$2\,A\,x_1 + A\,x_2 = \begin{pmatrix}5\\ 10\end{pmatrix}$$

$$2\,x_1 + x_2 = \begin{pmatrix}3\\ -1\end{pmatrix}$$

$$A\,(2\,x_1 + x_2) = \begin{pmatrix}2 & 1\\ 3 & -1\end{pmatrix}\begin{pmatrix}3\\ -1\end{pmatrix} = \begin{pmatrix}2\cdot 3+1\cdot(-1)\\ 3\cdot 3-1\cdot(-1)\end{pmatrix} = \begin{pmatrix}5\\ 10\end{pmatrix}$$

3.

(i)

$$\sigma_x\,e_1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = \begin{pmatrix}0\cdot 1+1.0\\ 1\cdot 1+0\cdot 0\end{pmatrix} = \begin{pmatrix}0\\ 1\end{pmatrix} = e_2$$

$$\sigma_x\,e_2 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix} = \begin{pmatrix}0\cdot 0+1.1\\ 1\cdot 0+0\cdot 1\end{pmatrix} = \begin{pmatrix}1\\ 0\end{pmatrix} = e_1$$

$$\sigma_y\,e_1 = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = \begin{pmatrix}0\cdot 1+-i\cdot 0\\ i\cdot 1+0\cdot 0\end{pmatrix} = \begin{pmatrix}0\\ i\end{pmatrix} = i\,e_2$$

$$\sigma_y\, e_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0\cdot 0 + -i\,.1 \\ 1\cdot 0 + 0\cdot 1 \end{pmatrix} = -i\begin{pmatrix} 1 \\ 0 \end{pmatrix} = -i\, e_1$$

$$\sigma_z\, e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1\cdot 1 + 0\cdot 0 \\ 0\cdot 1 + -1\cdot 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = e_1$$

$$\sigma_z\, e_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0\cdot 0 + 0.1 \\ 0\cdot 0 + -1\cdot 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = -\, e_2$$

(ii) $y = \sigma_z\, x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1\cdot 1 + 0\cdot 1 \\ 0\cdot 1 + -1\cdot 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\sigma_x\, \mathrm{ey} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0\cdot 1 + 1.(-1) \\ 1\cdot 1 + 0\cdot(-1) \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

4. determinants

$\det(A) = 1\cdot(-1) - 2\cdot 3 = -7,$
$\det(B) = 1\cdot 0 - c\cdot c = -c^2,$
$\det(C) = z\cdot z^* - 1 = \,|\,z\,|^2 - 1.$

5. Projectors

(i) $P_x\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$, $P_y\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ y \end{pmatrix}$. These are thus projectors on the x and y-axes, respectively.

(ii) Both of these give $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

6. Reflections

$$\sigma_x\, e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0\cdot 1 + 1.0 \\ 1\cdot 1 + 0\cdot 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = e_2$$

$$\sigma_x\, e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0\cdot 0 + 1.1 \\ 1\cdot 0 + 0\cdot 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = e_1$$

$$\sigma_x\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0\cdot 1 + 1.1 \\ 1\cdot 1 + 0\cdot 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Thus this swaps x and y components of any vector, and leaves one with $x = y$ invariant. That is a reflection in $y = x$.

7. Rotations

(i) Done in class. Check your notes!

(ii) $R(\theta)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos(\theta)\, x - \sin(\theta)\, y \\ \sin(\theta)\, x + \cos(\theta)\, y \end{pmatrix} = \pm\begin{pmatrix} x \\ y \end{pmatrix}$.

We can solve the simultaneous equations, but clearly $\sin(\theta) = 0$, $\cos(\theta) = \pm 1$, and thus $\theta = n\,\pi$, $n \in \mathbb{Z}$.

(iii) Using the prescription, $\frac{d}{d\theta} R(\theta) = \begin{pmatrix} -\sin(\theta) & -\cos(\theta) \\ \cos(\theta) & -\sin(\theta) \end{pmatrix}$, and for $\theta = 0$ we find that $X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Thus $R(\theta) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \theta\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + O\left(\theta^2\right)$.

(iv)$\left(\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}+\theta\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\right)\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}1\\\theta\end{pmatrix}$, $\left(\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}+\theta\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\right)\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}-\theta\\1\end{pmatrix}$. This approximation does not preserve length, and the angle is also not correct. It stil satisfies orthogonality, though. Make a sketch for your own use...