Solutions to Sheet 8

1. Sums

(i) $S_1 = 2\sum_{n=1}^{50} n = 2 \times \frac{1}{2}\, 50 \times 51 = 2550$

(ii) $S_2 = \sum_{n=1}^{10} 2^n = \frac{1-2^{11}}{1-2} = 2^{11} - 1 = 2047$

(iii) $\binom{4}{2} = \frac{4!}{2!^2} = \frac{24}{4} = 6$

$\binom{n}{n} = \frac{n!}{n!\,0!} = 1$

$\binom{n}{0} = \frac{n!}{0!\,n!} = 1$

(iv) For $n = 1$ $\sum_{k=1}^{1} 2\,k - 1 = 1 = 1^2$ ☺

If $\sum_{k=1}^{n} 2\,k - 1 = n^2$, then $\sum_{k=1}^{n+1} 2\,k - 1 = \sum_{k=1}^{n} 2\,k - 1 + 2\,n + 1 = n^2 + 2\,n + 1 = (n+1)^2$. ☺

Thus we can now use the induction bootstrap to proof validity for all n.

2. More sums

(i) $S_1 = \sum_{n=1}^{\infty} \frac{1}{4^n} = \frac{1}{1-1/4} = \frac{4}{3}$.

(ii) $S_2 = \sum_{n=1}^{\infty} \left(-\frac{1}{2}\right)^n = \frac{1}{1-(-1/2)} = \frac{2}{3}$.

(iii) $S_3 = \sum_{n=1}^{\infty} (a\,b)^n = \frac{1}{1-a\,b}$. If $|a\,b| < 1$.

3. Yet more sums

(i) $S_1 = \sum_{k=0}^{n} a_k - \sum_{m=0}^{n+2} a_m = \sum_{k=0}^{n} a_k - \sum_{k=0}^{n+2} a_k = -\sum_{k=n+1}^{n+2} a_k = -a_{n+1} - a_{n+2}$.

(ii) $S_2 = \sum_{k=m}^{n} a_k - \sum_{k=m+1}^{n+2} a_{k-1} = \sum_{k=m}^{n} a_k - \sum_{l=m}^{n+1} a_l = -a_{n+1}$.

3. Math Problems

(i) $f(0) = 1$;

$f\,'(x) = 2\exp(2\,x) \Longrightarrow f\,'(0) = 2$;

$f\,''(x) = 4\exp(2\,x) \Longrightarrow f\,''(0) = 4$;

$f(x) = \sum_{k=0}^{\infty} \frac{2^k}{k!}\, x^k$

(ii) $f(0) = 0$;

$f\,'(x) = 4\cos(4\,x) \Longrightarrow f\,'(0) = 4$;

$f\,''(x) = -4^2 \sin(4\,x) \Longrightarrow f\,''(0) = 0$;

$f^{(3)}(x) = -4^3\cos(4\,x) \Longrightarrow f^{(3)}(0) = -4^3; f^{(4)}(x) = 4^4\sin(4\,x) \Longrightarrow f^{(4)}(0) = 0$;

$f(x) = \sum_{k=0}^{\infty} \frac{1}{2\,k+1!}\, (-4\,x)^{2\,k+1}$

(iii) $f(0) = 1$;

$f'(x) = -4\sin(4x) \Longrightarrow f'(0) = 0$;

$f''(x) = -4^2\cos(4x) \Longrightarrow f''(0) = -4^2$;

$f^{(3)}(x) = 4^3\sin(4x) \Longrightarrow f^{(3)}(0) = 0; f^{(4)}(x) = 4^4\cos(4x) \Longrightarrow f^{(4)}(0) = 4^2$;

$f(x) = \sum_{k=0}^{\infty} \frac{1}{2k!}(-4x)^{2k}$

5.

Clearly, for $n = 1$, $(1+x)^1 = (1+x)$ ☺.

If it is true for n, $(1+x)^{n+1} = (1+x)(1+x)^n \geq (1+x)(1+nx)$

(provided that $1 + x \geq 0$, $x \geq -1$) $(1+x)(1+nx) = 1 + (n+1)x + nx^2 \geq 1 + (n+1)x$.

This proves the induction step. ☺

6.

(i) Up to you

(ii) First step: $(1-R)$ transmitted to the right, R refelected. Second step $R(1-R)$ of the initial intensity transmitted to the left, R^2 reflected. Third Step: $(1-R)R^2$ transmitted to right, R^3 reflected.

Conclusion: Transmitted to the right:

$T_R = (1-R) + (1-R)R^2 + (1-R)R^4 = \frac{1-R}{1-R^2} = \frac{1}{1+R}$.

Transmitted to the left:

$T_L = (1-R)R + (1-R)R^3 + (1-R)R^5 = R\,\frac{1-R}{1-R^2} = \frac{R}{1+R}$.

These add up to 1 (i.e., all light escapes)

6.

(i) Mass of nth ($n = 1, 2$) block is $2M/2^n$ (with M the mass of the first block). The centre-of-mass of that block is at $\left(\frac{1}{2^n}, \left(n - \frac{1}{2}\right)h\right)$

$Y_N = \frac{1}{M_N}\sum_{n=1}^{N} 2\,\frac{M}{2^n}\left(n - \frac{1}{2}\right)h$

$X_N = \frac{1}{M_N}\sum_{n=1}^{N} 2\,\frac{M}{2^n}\,\frac{1}{2^n}$

$M_N = \sum_{n=1}^{N} 2\,\frac{M}{2^n}$

We find that

$M_N = M\,\frac{\left(1-\frac{1}{2}^N\right)}{\left(1-\frac{1}{2}\right)} = M\,2\left(1-\left(\frac{1}{2}\right)^N\right)$.

Similarly,

$X_N = \frac{1}{M_N}\sum_{n=1}^{N} 2\,\frac{M}{2^n}\,\frac{1}{2^n} = \frac{M}{M_N}\,\frac{1}{2}\,\frac{\left(1-\frac{1}{4}^N\right)}{\left(1-\frac{1}{4}\right)} = \frac{1}{3}\,\frac{\left(1-\left(\frac{1}{4}\right)^N\right)}{\left(1-\left(\frac{1}{2}\right)^N\right)}$

(ii) The limit for $N \to \infty$ is now easy: $X_\infty = \frac{1}{3}$.

Note $Y_N = \frac{1}{M_N}2^{-N}\,h\,M\left(-2N + 3 \times 2^N - 3\right)$, $\lim_{N\to\infty} Y_N = \frac{3}{2}\,h$.