Solutions to Sheet 4

1. Ball on track, friction

(i) Newtons law, $F = m\,a$ turns into $m\,\dot{v} = -\gamma\,v$or $m\,\ddot{x} = -\gamma\,\dot{x}$.

(ii) Solution of $\dot{v} = \frac{-\gamma}{m}\,v$ is $\dot{x} = v = A\exp\left(\frac{-\gamma}{m}\,t\right)$, and thus $x = x_0 + B\exp\left(\frac{-\gamma}{m}\,t\right)$, where $B = \frac{-m}{\gamma}\,A$. Using $v_0 = A$,
$x = x_0 - \frac{m}{\gamma}\,v_0\exp\left(\frac{-\gamma}{m}\,t\right)$.

2. Ball on track, external force

(i) $m\,\ddot{x} = -c$, linear

(ii) $m\,\ddot{x} = -k\,x$, linear

(iii) $m\,\ddot{x} = -V_0\cos(x)$, nonlinear

(ii) $m\,\ddot{x} = -n\,\alpha\,x^{n-1}$, linear for n=0, n=1 and n=2

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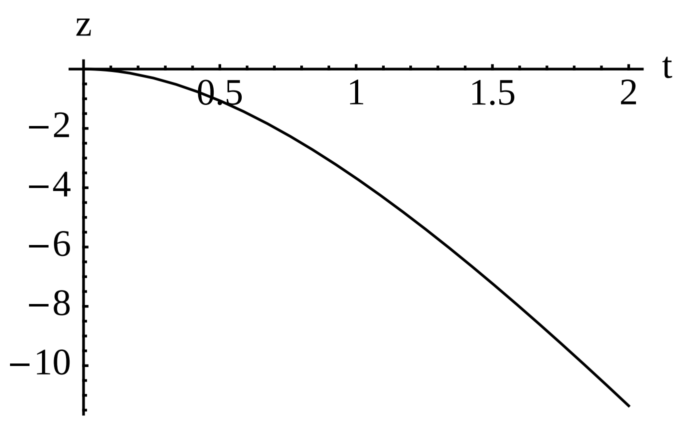
(i) $m\,\ddot{z} = -m\,g - \gamma\,\dot{z}$.

(ii) Inhomogeneous, but linear and first order, $\dot{w} = -g - \frac{\gamma}{m}\,w$.
Use integrating factor, $\frac{d}{dt}\left(\exp\left(\frac{\gamma}{m}\,t\right)w\right) = -g\exp\left(\frac{\gamma}{m}\,t\right)$,
$w = e^{-\frac{t\gamma}{m}}\,A - \frac{g\,m}{\gamma}$. Use initial conditions, $w(0) = 0 = A - g\,\frac{m}{\gamma}$. Integrate one more time,
$z = -g\,\frac{m^2}{\gamma^2}\left(e^{-\frac{t\gamma}{m}} - 1\right) - \frac{g\,m}{\gamma}\,t$

Plot$\left[-g\,\frac{m^2}{\gamma^2}\left(e^{-\frac{t\gamma}{m}} - 1\right) - \frac{g\,m}{\gamma}\,t\ /.\ \{g \to 10,\ m \to 1,\ \gamma \to 1\},\ \{t,\ 0,\ 2\},\ \text{AxesLabel} \to \{t,\ z\}\right]$



- Graphics -

(iii) As $t \to \infty$ we have $w = -g \frac{m}{\gamma}$, which corresponds to a balance of forces.

4.

(i) $(e^{\alpha x})' = \alpha e^{\alpha x}$: $\left(\alpha^2 - \alpha - 2\right) e^{\alpha x} = 0$ or $\alpha = \frac{1 \pm \sqrt{1+8}}{2} = -1, 2$. Thus $y(x) = e^{-x}$ and $y(x) = e^{2x}$ are both solutions.
(ii) differentiales once:
$y' = -c_1 e^{-x} + 2 c_2 e^{2x}$, and twice
$y'' = c_1 e^{-x} + 4 c_2 e^{2x}$. Add together:
$y''(x) - y'(x) - 2 y(x) = c_1(1 - (-1) - 2) e^{-x} + c_2(4 - 2 - 2) e^{2x} = 0$
(iii) $y(0) = c_1 + c_2 = 0$, $y'(0) = -c_1 + 2 c_2 = 1 \Rightarrow c_1 = -1/3$, $c_2 = 1/3$:
$y(x) = \frac{1}{3} \left(-e^{-x} + e^{2x}\right)$.

5.

(i) $\left(e^{ix}\right)'' = -e^{ix}$, $\left(e^{-ix}\right)'' = -e^{-ix}$.
(ii) $\sin(x)'' = -\sin(x)$, $\cos(x)'' = -\cos(x)$.
(iii) $v_1(x) = \frac{1}{2i} (y_1 - y_2)$, $v_2(x) = \frac{1}{2} (y_1 + y_2)$.
(iv) $y(0) = c_2 = 1$, $y'(0) = c_1 = 0$: $y(x) = \sin(x)$.

6.

(i) $(e^{x})'' = e^{x}$, $(e^{-x})'' = -e^{-x}$.
(ii) $\sinh(x)'' = \sinh(x)$, $\cosh(x)'' = \cosh(x)$.
(iii) $v_1(x) = \frac{1}{2} (y_1 - y_2)$, $v_2(x) = \frac{1}{2} (y_1 + y_2)$.
(iv) $y(0) = c_2 = 0$, $y'(0) = c_1 = 1$: $y(x) = \cosh(x)$.

7.

(i) $(e^{\alpha x})' = \alpha\, e^{\alpha x}$, $(e^{\alpha x})'' = \alpha^2\, e^{\alpha x} \Rightarrow \left(\alpha^2 - 4\alpha + 5\right) e^{\alpha x} = 0$, $\alpha = 2 \pm i$
(ii) $y(x) = c_1 e^{(2+i)x} + c_2 e^{(2-i)x} = e^{2x}\left(c_1 e^{ix} + c_2 e^{-ix}\right) =$
$e^{2x}(c_1(\cos(x) + i \sin(x)) + c_2(\cos(x) - i \sin(x))) = e^{2x}((c_1 + c_2) \cos(x) + i(c_1 - c_2) \sin(x))$