Solutions to Sheet 12

Notations: Det for det, . for matrix/matrix product, Inverse[] for power -1.

1.

$$z = \frac{10 \pm \sqrt{10^2 - 4 \cdot 40}}{2} = 5 \pm \sqrt{-15} = 5 \pm i \sqrt{15}\,.$$

2.

$$\frac{1}{x+i} = \frac{1}{x+i}\,\frac{x-i}{x-i} = \frac{x-i}{x^2+1} = \frac{x}{x^2+1} + i\,\frac{-1}{x^2+1}$$

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Needs["PlotLegends`"]
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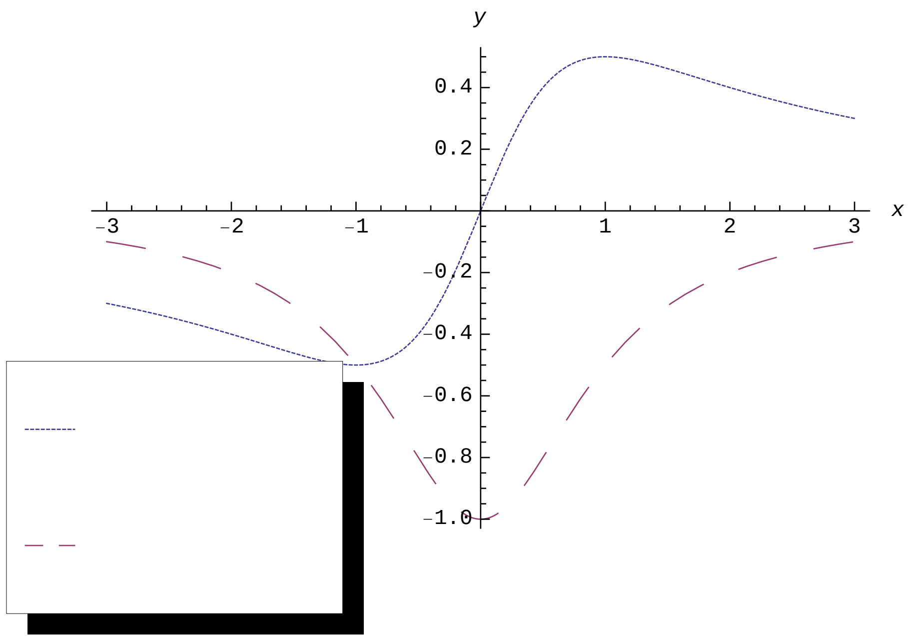
$$\text{Plot}\left[\left\{\frac{x}{x^2+1}, -\frac{1}{x^2+1}\right\}, \{x, -3, 3\}, \text{PlotStyle} \to \{\text{Dashing}[\{0.\}], \text{Dashing}[\{0.05\}]\}, \text{PlotLegend} \to \{\text{"Re"}, \text{"Im"}\}, \text{AxesLabel} \to \{x, y\}\right]$$



3.

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Show[Graphics[{GrayLevel[0.6`], Disk[{0, 0}, 2]},
   Method → {"AxesInFront" → True}, Axes → True],
  Graphics[{GrayLevel[1.`], Disk[{0, 0}, 1]},
   Method → {"AxesInFront" → True}, Axes → True],
  AspectRatio → 1, AxesLabel → {Re[z], Im[z]}]
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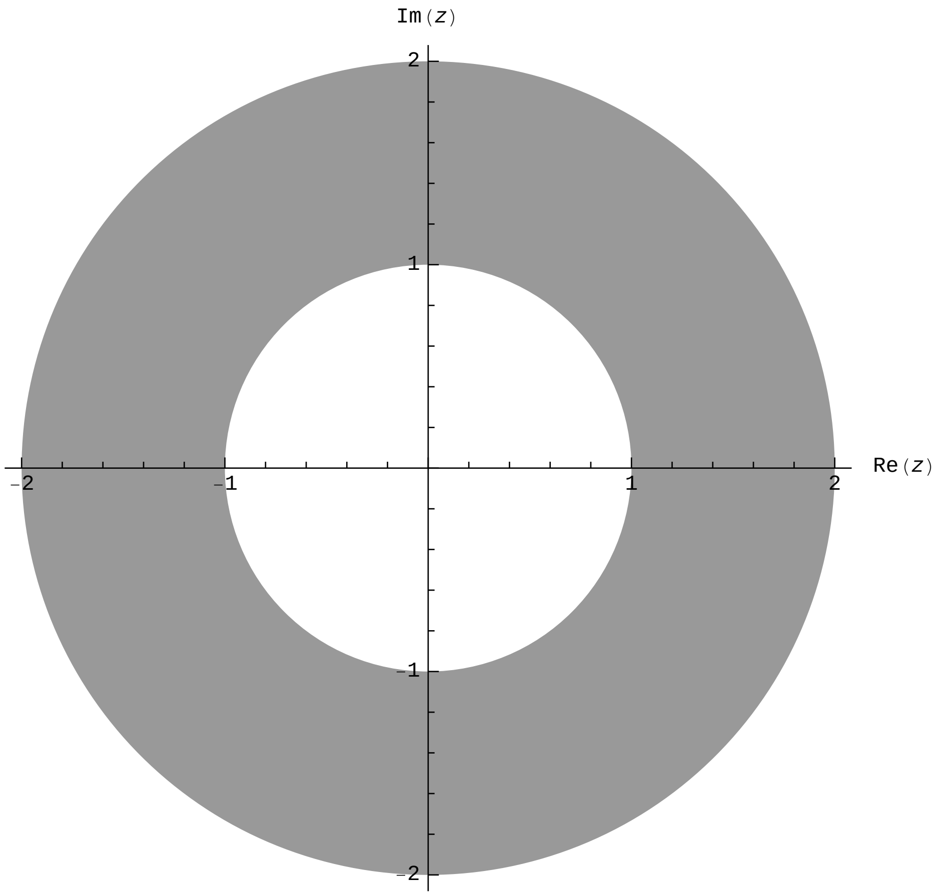


4.

$e^{i(\omega+i\gamma)t} = e^{-\gamma t}(\cos(\omega t) + i\sin(\omega t))$, $\mathrm{Re}\left(e^{i\Omega t}\right) = e^{-\gamma t}\cos(\omega t)$

5.

$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $\cos(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2m!}$, $\sin(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{2m+1!}$.

6.

$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} \Rightarrow \tanh'(x) = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x}$.

7.

$y'(x) = e^{px}$: By direct integration $y(x) = \frac{1}{p} e^{px} + c$. y(0)=1 $\Rightarrow \frac{1}{p} + c = 0 \Rightarrow c = \frac{-1}{p}$: $y(x) = \frac{1}{p}(e^{px} - 1)$

8.

Substitute $y(x) = e^{z x}$. Find $\left(z^2 - z - 2\right) e^{z x} = 0$. Conclude $z = \frac{1 \pm \sqrt{1+8}}{2} = -1, 2$. Thus $y(x) = A\, e^{-x} + B\, e^{2 x}$. $y(0) = 0 \Rightarrow A + B = 0$; $y'\,(0) = 1 \Rightarrow -A + 2\,B = 1$. Solve the simulateneous equations and find that $B = \frac{1}{3}$, $A = -\frac{1}{3}$. Thus $y(x) = \frac{-1}{3}\, e^{-x} + \frac{1}{3}\, e^{2 x}$.

9.

$y(x) = A \cos\left(\sqrt{a}\; x\right) + B \sin\left(\sqrt{a}\; x\right)$.

10.

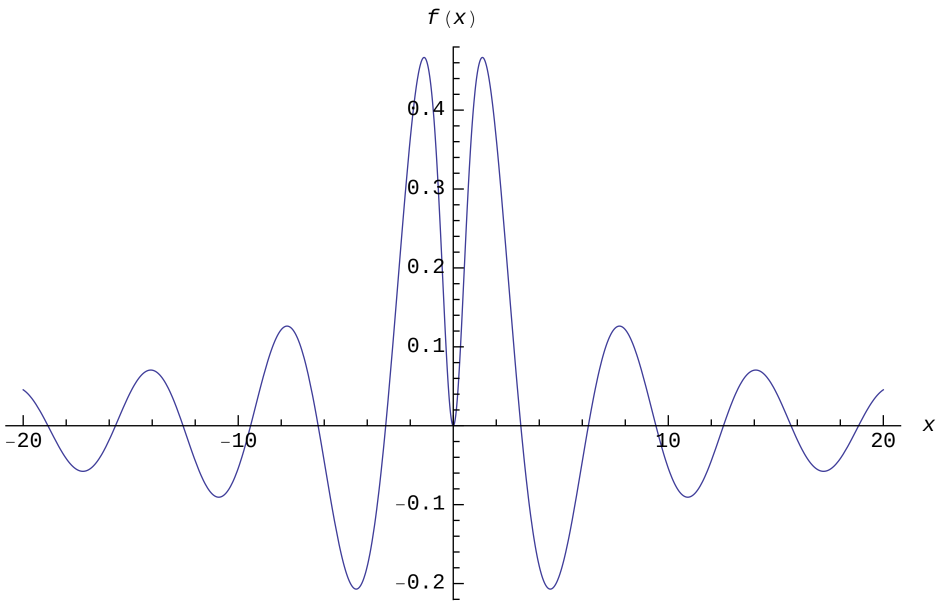
(i) Substitute $y(x) = e^{z x}$. Find $\left(z^2 - 4 z + 5\right) e^{z x} = 0$. Conclude $z = \frac{4 \pm \sqrt{16-20}}{2} = 2 \pm 2\, i$. Thus $y(x) = A\, e^{(2-2 i) x} + B\, e^{(2+2 i) x} = e^{2 x}(C \cos(2\, x) + D\, \mathrm{isn}(2\, x))$.

(ii) Substitute $y(x) = e^{z x}$. Find $\left(z^2 + 2 z + 1\right) e^{z x} = 0$. Conclude $z = -1$. Special case (two equal roots, critical damping): $y(x) = (A + B\, x)\, e^{-x}$.

(iii) Substitute $y(x) = e^{z x}$. Find $\left(z^2 - a\right) e^{z x} = 0$. Conclude $z = \pm \sqrt{a}\;\; (\in \mathbb{R})$. Thus $y(x) = A\, e^{-\sqrt{a}\; x} + B\, e^{\sqrt{a}\; x}$.

11.

Plot[$x \sin(x) / \left(1 + x^2\right)$, {$x$, −20, 20}, AxesLabel → {x, $f(x)$}]



12.

$\sum_{n=0}^{\infty} (2\, a)^n = \frac{1}{1-2\, a}$ (geometric series). This converges is $|\, 2\, a\, | < 1$, or $\left| a \right| < \frac{1}{2}$.

13.

(i)

$f(x) \quad (2+x)^{1/2} \qquad f(0) \quad \sqrt{2}$

$f^{(1)}(x) \quad \frac{1}{2}(2+x)^{-1/2} \qquad f^{(1)}(0) \quad \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$

$f^{(2)}(x) \quad \frac{-1}{4}\,(2+x)^{-3/2} \qquad f(0) \quad -\frac{1}{4 \cdot 2^{-3/2}} = \frac{-\sqrt{2}}{16}$

$f(x) = \sqrt{2} + \frac{\sqrt{2}}{4}\,x - \frac{\sqrt{2}}{32}\,x^2 + O\left(x^3\right)$

(ii)

$\sin(x)/x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}\,\frac{1}{x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n+1)!} = 1 + \frac{x^2}{6} + O\left(x^4\right)$

14.

(i) $e^{4+x^2} = e^4\,e^{x^2} = e^4 \sum_{m=0}^{\infty} \frac{\left(x^2\right)^m}{m!} = \sum_{m=0}^{\infty} \frac{e^4\,x^{2m}}{m!}$

(ii) $\frac{1}{1+x^6} = \sum_{n=0}^{\infty} \left(-x^6\right)^n = \sum_{n=0}^{\infty} (-1)^n\,x^{6n}$

15.

$\sqrt{100-a} = 10\,\sqrt{1-a/100} = 10 \sum_{n=0}^{\infty} \left(\frac{a}{100}\right)^n = 10\left(1 + \frac{a}{100} + \frac{a^2}{10\,000}\right)$. Neglected term: $\frac{a^3}{10^6}$is less than 10^{-3}; accuracy is thus of the order 10^{-2}, i.e., two decimal places.

16.

$\sin\left(a\,x^2\right) = a\,x^2 + O\left(x^4\right)$; $\frac{x^2}{\sin\left(a\,x^2\right)} = \frac{x^2}{a\,x^2 + O\left(x^4\right)} = \frac{1}{a + O\left(x^2\right)}$. Thus $\lim_{x\to 0} \frac{x^2}{\sin\left(a\,x^2\right)} = \frac{1}{a}$.

17.

$\frac{\partial}{\partial x} \sin\left(x + x^2\,y^3\right) = \cos\left(x + x^2\,y^3\right) \frac{\partial}{\partial x}\left(x + x^2\,y^3\right) = \cos\left(x + x^2\,y^3\right)\left(1 + 2\,x\,y^3\right)$

$\frac{\partial}{\partial y}\left(\cos\left(x + x^2\,y^3\right)\left(1 + 2\,x\,y^3\right)\right) =$

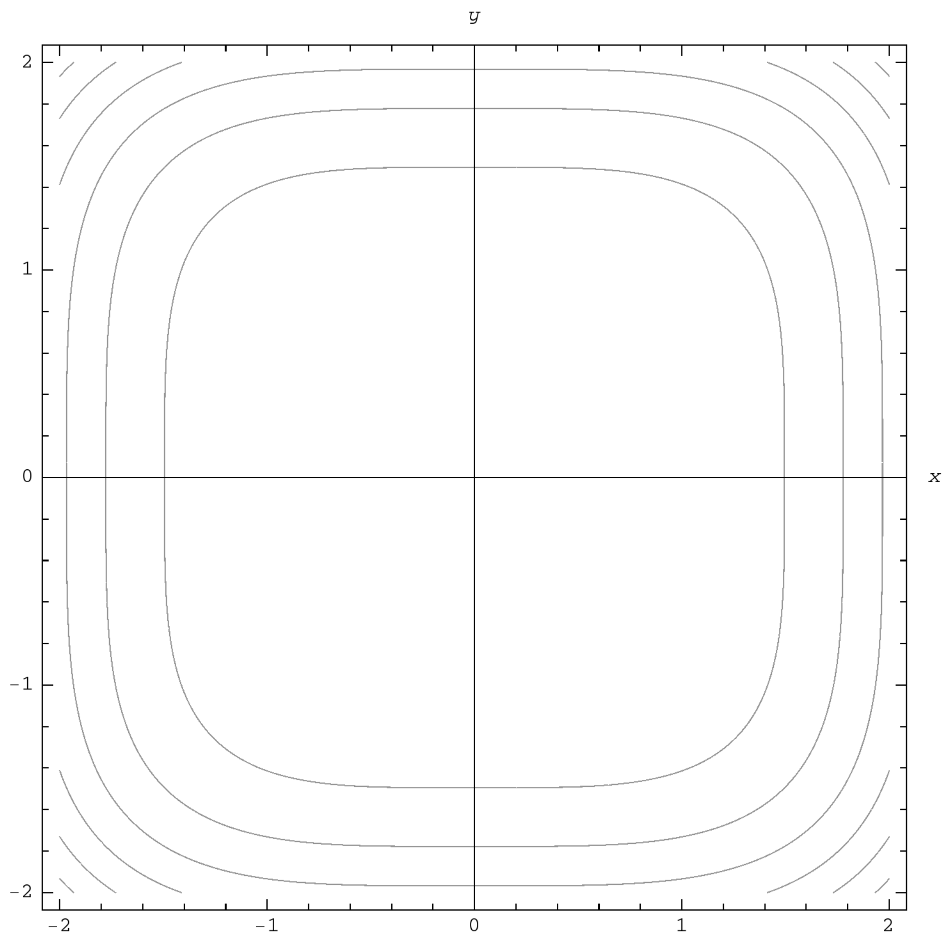
$\quad -\sin\left(x + x^2\,y^3\right) \frac{\partial}{\partial y}\left(x + x^2\,y^3\right)\left(1 + 2\,x\,y^3\right) + \cos\left(x + x^2\,y^3\right) \frac{\partial}{\partial y}\left(1 + 2\,x\,y^3\right) =$

$\qquad -\sin\left(x + x^2\,y^3\right) x^2\,3\,y^2\left(1 + 2\,x\,y^3\right) + \cos\left(x + x^2\,y^3\right) 6\,x\,y^2$

18.

(i)

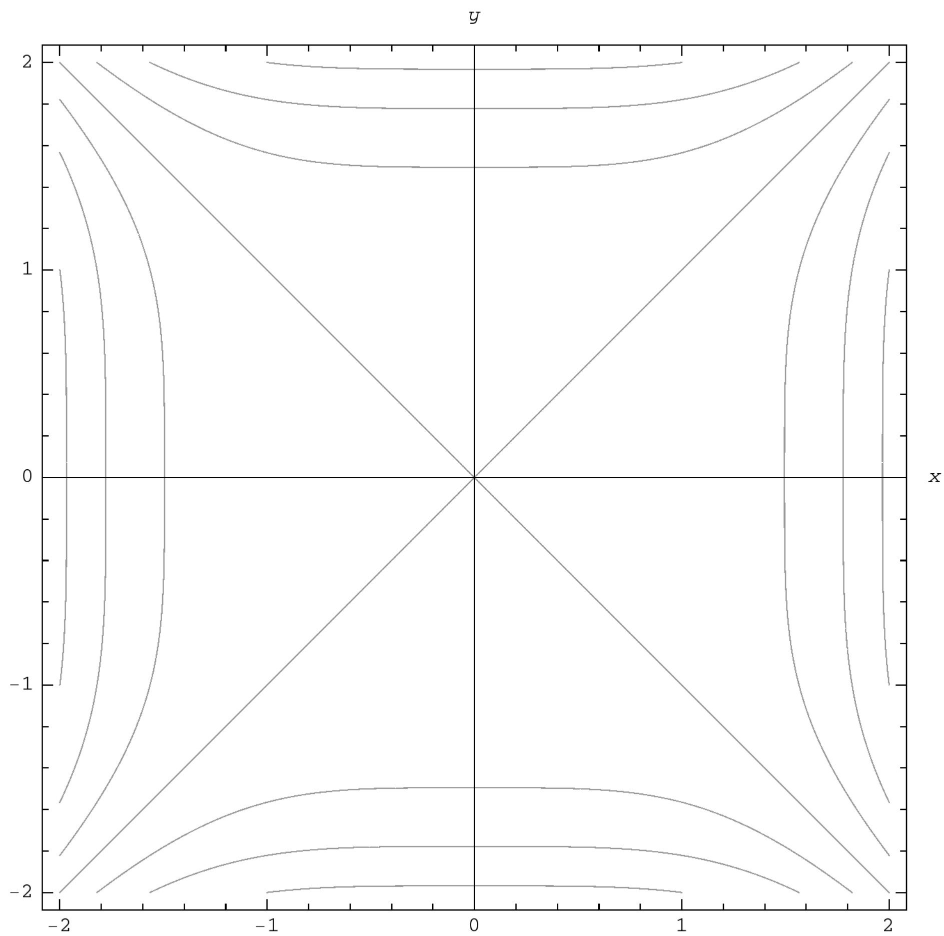
**ContourPlot[$(x^4 + y^4)$, {x, −2, 2}, {y, −2, 2}, Axes −> True,
AxesLabel −> {x, y}, ContourShading −> False, PlotPoints → 100]**



grad $(x^4 + y^4) = (4x^3, 4y^3)$. This is zero when $x = 0 \wedge y = 0$. This is obviously a minimum.

(ii)

**ContourPlot[$(x^4 - y^4)$, {x, −2, 2}, {y, −2, 2}, Axes −> True,
AxesLabel −> {x, y}, ContourShading −> False, PlotPoints → 100]**



grad $(x^4 - y^4) = (4x^3, -4y^3)$. This is zero when $x = 0 \wedge y = 0$. This is obviously a saddlepoint.

19.

(i) $\det(B) = 1 \cdot 0 - c \cdot c = -c^2$; $\det(C) = z \cdot z^* - 1 \cdot 1 = |z|^2 - 1$.

(ii) $BC = \begin{pmatrix} 1 & c \\ c & 0 \end{pmatrix}\begin{pmatrix} z & 1 \\ 1 & z^* \end{pmatrix} = \begin{pmatrix} 1 \cdot z + c \cdot 1 & 1 \cdot 1 + c \cdot z^* \\ c \cdot z + 0 \cdot 1 & c \cdot 1 + 0 \cdot z^* \end{pmatrix} = \begin{pmatrix} z + c & 1 + c\,z^* \\ c\,z & c \end{pmatrix}$

(iii) $C^{-1} = \frac{1}{|z|^2 - 1}\begin{pmatrix} z^* & -1 \\ -1 & z \end{pmatrix}$ (check $C\,C^{-1} = I$!)

(iv)

$\det(B - \lambda I) = 0 \Rightarrow \begin{vmatrix} 1 - \lambda & c \\ c & -\lambda \end{vmatrix} = -\lambda(1 - \lambda) - c^2 = \lambda^2 - \lambda - c^2 = 0.$

Thus $\lambda = \frac{1 \pm \sqrt{1 + 4c^2}}{2}$.

$\det(C - \lambda I) =$.

$$0 \Rightarrow \begin{vmatrix} z^* - \lambda & 1 \\ 1 & z - \lambda \end{vmatrix} = (z^* - \lambda)(z - \lambda) - 1 = \lambda^2 - \lambda(z + z^*) + |z|^2 - 1 = \lambda^2 - 2\,\mathrm{Re}(z)\,\lambda + |z|^2 - 1 = 0$$

Thus $\lambda = \mathrm{Re}(z) \pm \sqrt{\mathrm{Re}(z)^2 - |z|^2 + 1} = \mathrm{Re}(z) \pm \sqrt{1 - \mathrm{Im}(z)^2}$.