PHYS 30672 MATHS METHODS Examples 3

1. Mirages occur where the refractive index of the air $n(z)$ increases with height z above the surface. According to Fermat's principle, the path of light travelling from A to B is the one minimising the travel time between these points.

(i) Show that for this path,

$$\frac{dz}{dx} = \sqrt{A^2 n(z)^2 - 1}\ .$$

(ii) If $n(z) = n_0(1 + \alpha z)$, show that

$$An(z) = \cosh\left(An_0\alpha[x - x_0]\right)$$

where x_0 is a constant.
(iii) For a ray just grazing the surface at $x = 0$ show that

$$1 + \alpha z = \cosh(\alpha x).$$

(iv) Assuming αx is small, show that for an observation point P at height z, the grazing ray appears to come from a point at distance $d = \sqrt{\frac{z}{2\alpha}}$.

2. A soap film is formed between two horizontal rings, one vertically above the other.

The surface of the film $r(z)$ is determined by minimizing the static energy, comprising the surface energy $= \sigma\times$ Area, plus the gravitational energy, assuming constant density ρ per unit area. Derive the differential equation

$$(\sigma + g\rho z)\ \sqrt{1 + r'^2} = \frac{d}{dz}\left((\sigma + g\rho z)\ \frac{rr'}{\sqrt{1 + r'^2}}\right)$$

but don't try solving it.

3. Using polar coordinates (R, θ, ϕ) on the surface of a sphere of radius R, show that the shortest path $\theta(\phi)$ joining two points on the surface satisfies the equation

$$\left(\frac{d\theta}{d\phi}\right)^2 = A\sin^4\theta - \sin^2\theta,$$

where A is constant.

[This equation is hard to solve, but there is an analytic solution, which of course corresponds to a *great circle* path round the sphere.]

4. In relativistic quantum mechanics, the Lagrangian density for a neutral π-meson is given by

$$\mathcal{L}(\boldsymbol{x}, t) = \frac{1}{2}\left(\dot{\phi}^2 - \nabla\phi.\nabla\phi - \mu^2\phi^2\right)\ ,$$

where μ is the pion mass and $\phi(\boldsymbol{x}, t)$ is a real wavefunction. Assuming Hamilton's principle $\delta S = 0$, where the "action"

$$S = \int d^3x\, dt\, \mathcal{L}(\boldsymbol{x}, t)\ ,$$

find the wave equation satisfied by ϕ.

5. A solid spherical planet of radius R rotates with angular velocity ω, and is covered with a layer of water of depth $h(\theta) \ll R$, where θ is the polar angle.

For a small volume of water dV in the layer, at position $y(\theta)$ above the solid surface, $0 < y(\theta) < h(\theta)$, the gravitational potential energy is $g\rho\, y\, dV$ and the rotational potential energy is $-\rho\omega^2 R^2 \sin^2\theta\, dV/2$, where ρ is the density of water, g is the acceleration due to gravity and the approximation $y \ll R$ has been used. In the same approximation, the volume dV is $dV = R^2 \sin\theta\, dy\, d\theta\, d\phi$.

Show that the total potential energy of the water is

$$E = \pi\rho R^2 \int_0^\pi \left(gh^2 - \omega^2 R^2 h \sin^2\theta\right) \sin\theta \mathrm{d}\theta\ .$$

Minimise E subject to the condition of constant volume V and hence derive an expression for the depth of water $h(\theta)$.

This was the second part of an exam question set in June 1999.

6. The Hamiltonian operator for an electron particle constrained to the region $0 < x < \infty$ is

$$H = \left(-\frac{d^2}{dx^2} + x\right),$$

where units have been chosen such that $\frac{\hbar^2}{2m}$. The boundary conditions for the wave function are:

$$\Psi \to 0 \text{ as } x \to \infty;$$

$$\frac{\mathrm{d}\Psi}{\mathrm{d}x} = 0 \text{ at } x = 0\,.$$

Choosing the trial function $\Psi = (\beta x + 1)\exp(-\alpha x)$, where $\alpha,\ \beta$ are real constants, determine a value for β such that the boundary conditions are satisfied. Show that for this trial function,

$$\int_0^\infty \Psi^* H \Psi \, dx = \frac{\alpha}{4} + \frac{9}{8\alpha^2}\,.$$

Use the Rayleigh-Ritz method to estimate a value for the ground state energy.

You may assume the integral

$$\int_0^\infty x^n \exp(-\gamma x)\, dx = \frac{n!}{\gamma^{n+1}}.$$