Workshop 11, Week 11

Please follow the instructions of your supervisor regarding timing of these problems.
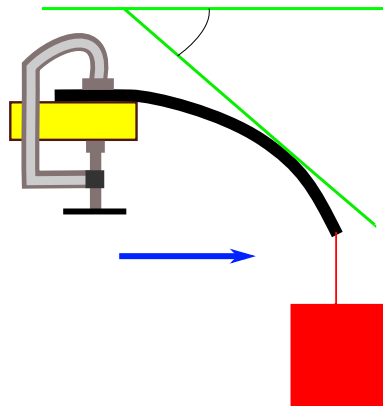
Math Review

1. $*$ Given that $\sin'(x) = \cos(x)$, $\cos'(x) = -\sin(x)$, $\ln'(x) = 1/x$, differentiate:

    (i) $\sin(x)/x$,   (ii) $\tan(x)$,

    (iii) $\sin(x^2)$,   (iv) $x^2\ln(x^2)$.

Physics Problems

2. $*$ A (massless) flexible beam of length l is clamped at one end, and a mass is suspended from the other end.

The tangent at position x (the angle with the horizontal) is $\theta(x)$, and satisfies the differential equation

$$\cos\theta \frac{d\theta}{dx} = k(l - x),$$

with k a constant.

(i) Discuss the forces playing a role in this problem.
(ii) Find the general solution of the differential equation, as well as the one satisfying $\theta = 0$ at $x = 0$.
(iii) What relation must the values k and l satisfy, in order for the beam not to break?
(iv) Discuss the way to find the position as a function of x.

3. We study a particle in free-fall in the earth's gravitational field.
(i) Show from conservation of energy that if the particle is released at a distance a from the centre of the earth, its velocity at a distance r from the centre of the earth is

$$v = -\sqrt{2GM}\sqrt{-\frac{1}{a} + \frac{1}{r}} \quad .$$

(ii) Show that the differential equation for the time spend in free-fall between the points a and r is

$$\sqrt{\frac{2GM}{a}}\frac{dt}{dr} = \sqrt{\frac{r}{a - r}} \quad .$$

Express the length in terms of the radius of the earth, $a = \alpha R$, $r = xR$, and time as $\tau = \sqrt{2GMR}t$.
(iii) Solve the differential equation (**Hint:** consider the substitution $x = \alpha\sin^2(\phi)$).
(iv) Show that for near-earth free fall we get the relation $h = \frac{1}{2}gt^2$. Express g in G and M.

Maths Practice

In the notation used below, $y' = \frac{dy}{dx}$.

4. (i) $*$ Find the solution of $\dfrac{dz}{dt} = 1 + z^2$ that satisfies $z = 1$ at $t = 0$,
(ii) Find the general solution of $y' = \sin^2(y)$.
(iii) Find the solution of $y' = y^2 + 1$ satisfying the condition $y(0) = 1$.
(iv) $*$ Find the general solution of $(y')^2 = 1 - y^2$.

5. (i) $*$ Find the general solution of $(x - 1)(x - 2)y' = xy$,
(ii) Find the solution of $yy'\cos^2(x) = 2 + \tan(x)$ that satisfies $y = 2$ at $x = \pi/4$.

Reading for next week: Chapter 14 (derivatives)