P201 Workshop 4, Week 4

Please follow the instructions of your supervisor regarding timing of these problems.

Physics

1. A ball of mass m rolls on a track (x–axis), with the only force acting on it being a friction force $-\gamma v(t)$ ($v(t)$: velocity of the ball at time t.)

   (i) Write down Newton's law as a differential equation for the velocity $v(t)$.

   (ii) Solve this differential equation and determine the position $x(t)$ of the ball, assuming that a time $t = 0$, the position is $x(t = 0) = x_0$ and the velocity is $v(t = 0) = v_0$.

2. A particle of mass m moves on a line (x–axis) under the influence of a force $F(x)$, where x is the position of the particle. The function $V(x)$ with $F(x) = -V'(x)$ is called the potential of the force.

   Derive the equation of motion of the particle in the standard form $\ddot{x}(t) + ... = 0$ for the following potentials, and classify the equation as linear/non–linear:

   (i) $V(x) = cx, c > 0$; (ii) $V(x) = -(k/2)x^2, k > 0$; (iii) $V(x) = V_0 \sin(x)$

   * (iv) For which integer(s) n does the potential $V(x) = \alpha x^n$ lead to a linear equation of motion? Sketch $V(x)$ for positive and negative α for the largest n, and sketch typical curves $x = x(t)$.

3. Alice has falls into an infinitely deep well (along z–axis). She hass mass m, and the two forces acting on her are the gravity force $F_g = -mg, g > 0$, and the friction force $-\gamma w(t), \gamma > 0$, where $w(t)$ is the velocity of Alice at time t (all in the z–direction).

   (i) Write down Newton's law for this problem.

   * (ii) Solve the resulting differential equation. Assume that a time $t = 0$, the position of Alice is $z(t = 0) = 0$ and her velocity is $w(t = 0) = 0$.

   * (iii) Determine her 'stationary' velocity $w(t \to \infty)$ from the solution of the differential equation. Check that this is correct by balancing forces.

Math Practise + Problems

4. (i) Show that the differential equation $y''(x) - y'(x) - 2y(x) = 0$ has two solutions of the form $y(x) = e^{\alpha x}$, and determine the two allowed values for α. Hint: substitute $y = e^{\alpha x}$ into the differential equation.

   (ii) Show that $y(x) = c_1 e^{-x} + c_2 e^{2x}$ is the general solution of $y''(x) - y'(x) - 2y(x) = 0$.

   (iii) Determine c_1 and c_2 in 2 such that $y(0) = 0$ and $y'(0) = 1$.

5. Study the differential equation $y''(x) + y(x) = 0$

   (i) Check that this differential equation has two (linearly independent) solutions of the form $y_1(x) = e^{ix}$, $y_2(x) = e^{-ix}$.

   (ii) Check that this differential equation also has (linearly independent) solutions of the form $v_1(x) = \sin(x)$, $v_2(x) = \cos(x)$.

   (iii) Show that v_1 depends linearly on y_1 and y_2, and that v_2 depends linearly on y_1 and y_2 by expressing v_1 and v_2 as linear combinations of y_1 and y_2.

   (iv) Use the form $y(x) = c_1 \sin(x) + c_2 \cos(x)$ to solve this differential equation subject to the initial conditions $y(0) = 1, y'(0) = 0$.

6. Now look at the differential equation $y''(x) - y(x) = 0$

   (i) Show that this differential equation has two different solutions of the form $y_1(x) = e^x$, $y_2(x) = e^{-x}$.

   (ii) Show that this differential equation also has solutions of the form $v_1(x) = \sinh(x)$, $v_2(x) = \cosh(x)$.

   (iii) Show that one can express v_1 and v_2 as linear combinations of y_1 and y_2.

   (iv) Use the linear combination $c_1 \sinh(x) + c_2 \cosh(x)$ to solve this differential equation with the initial condition $y(0) = 0, y'(0) = 1$.

7. We solve the differential equation $y''(x) - 4y'(x) + 5y(x) = 0$

   (i) Show that this differential equation has two different solutions of the form $y_1(x) = e^{\alpha_1 x}$ and $y_2(x) = e^{\alpha_2 x}$ and determine the values α_1 and α_2.

   (ii) Write the general solution of this differential as $y(x) = c_1 y_1(x) + c_2 y_2(x)$ and show that this can be written as $y(x) = e^{2x}[d_1 \cos(x) + d_2 \sin(x)]$ with new constants d_1 and d_2.

Reading for next week: FM, Chap. 15