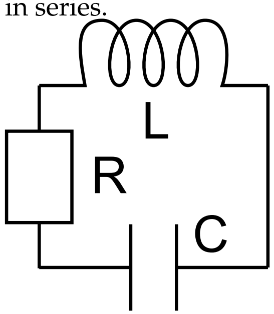
P201 Workshop 5, Week 5

Please follow the instructions of your supervisor regarding timing of these problems.

Physics

1. Consider a closed circuit with a resistor R, an inductor L, and a capacitor C in series.

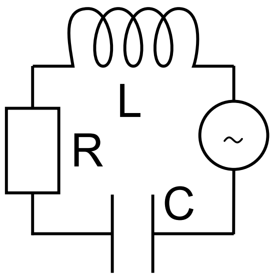


The time–dependent charge $Q(t)$ on the capacitor, the corresponding voltage drop across the capacitor $U(t)$, and the current $I(t)$ through the circuit are related by $Q(t) = CU(t)$ and $I(t) = -\dot{Q}(t)$. Furthermore, the voltage $U(t)$ must be equal to the sum of the voltages $L\dot{I}(t)$ (inductor) and $RI(t)$ (resistor), i.e., $U(t) = L\dot{I}(t) + RI(t)$

(i) Use these two equations to eliminate the charge $Q(t)$ and the current $I(t)$ in order to find a differential equation for $U(t)$. Show that this differential equation reads

$$\ddot{U}(t) + \frac{R}{L}\dot{U} + \frac{1}{LC}U(t) = 0. \tag{1}$$

(ii) Classify (1) (as nonlinear/linear, homogeneous or inhomogeneous) .
(iii) Find two independent solutions of (1) for the special case of zero resistance, $R = 0$ (insert $\exp(izt)$ and find possible values for z).
(iv) Show that the general solution can be written as $U(t) = [A\cos(\omega_0 t) + B\sin(\omega_0 t)]$ with $\omega_0 = \sqrt{1/LC}$.
(v) Now find two independent solutions of (1) for the general case, $R \neq 0$, (insert $\exp(izt)$ and find possible values for z). Assume that $R < 2\sqrt{L/C}$.
(vi) Show that the general solution can be written as $U(t) = e^{\alpha t}[A\cos(\omega t) + B\sin(\omega t)]$ with $\alpha = R/(2L), \omega = \sqrt{1/(LC) - R^2/(4L^2)}$.
(vii) Show that this solution is consistent with your solution in part 3.



Consider the same RLC circuit, but now with an external voltage $V(t) = V_0 \sin(\Omega t), V_0 > 0$ applied. The voltage $V(t)$ must fulfill $L\dot{I}(t) + RI(t) = Q(t)/C + V(t)$, where $I(t)$ is the time–dependent current through the circuit.

2. (i) Find the differential equation for $I(t) = -\dot{Q}(t)$ (hint: differentiate the equation for the voltage),

$$L\ddot{I}(t) + R\dot{I}(t) + \frac{1}{C}I(t) = \Omega V_0 \cos(\Omega t). \tag{2}$$

(ii) Classify (2) (as nonlinear/linear and homogeneous/inhomogeneous).
(iii) Consider the 'auxiliary' differential equation to (2), where $\cos(\Omega t) = \mathrm{Re}(e^{i\Omega t})$ is replaced by $e^{i\Omega t}$, $L\ddot{I}^a(t) + R\dot{I}^a(t) + I^a(t)/C = \Omega V_0 e^{i\Omega t}$, and solve this equation by inserting $I^a(t) = I_0 e^{i\Omega t}$ and determine I_0.
(iv) Calculate the polar form of the complex number $I_0 = |I_0|e^{i\phi}$ (note that $V_0 > 0$ is real!),
(v) * Prove the following: a) If the complex current $I^a(t)$ is a solution of the auxiliary equation, the complex conjugate $[I^a(t)]^*$ is a solution of the same equation, with $e^{i\Omega t}$ replaced by $e^{-i\Omega t}$, and b) consequently, the 'true' current $I(t) = \mathrm{Re}(I^a(t))$ is a solution of (2).
(vi) Use the polar form of I_0 to find the 'true' current $I(t) = \mathrm{Re}(I^a(t)) = \mathrm{Re}(I_0 e^{i\Omega t}) = |I_0|\cos(\Omega t + \phi)$, i.e. the solution of (2). What is the value of ϕ in the case $R = 0$ (zero resistor), and what is the phase shift between the current $I(t)$ and the voltage $U(t)$ in that case ?

Math Practise

3. Find the general solution of
(i) $y''(x) + 12y(x) = 0$; (ii) $y''(x) + y'(x) + 12y(x) = 0$.

4. Sketch the following curves
(i) $f(x) = 1/(1 + x^2)$, (ii) $f(x) = e^{-x^2}$, (iii) $f(x) = \sin(x)/(1 + x^2)$.

5. Find the polar form for the following complex numbers:
(i) $-i$, (ii) $1 - \sqrt{3}i$ (iii) $(1 + \sqrt{3}i)^{1/4}$.
(iv) Find all solutions to $z^4 = (-1 - \sqrt{3}i)$ (Hint: write in polar form.)

Math Problems

6. (i) Consider the differential equation $y''(x) + 4y(x) = 2e^{i3x}$. Write a particular solution $y_p(x)$ of this inhomogeneous equation as $y_p(x) = Ce^{i3x}$ and determine the constant C.

(ii) * Determine the general solution of this differential equation, and solve it for $y(0) = 1, y'(0) = 0$.

Reading for next week: FM, Chap. 15