Workshop 4, Week 4

Please follow the instructions of your supervisor regarding timing of these problems.

1. Ahow that the functions $f_1(x) = 1$ and $f_2(x) = (3x^2 - 1)/2$ are orthogonal on the interval $(-1, 1)$ (weight function $w(x) = 1$). Find the norm of these functions, and use orthogonality to decompose $f(x) = x^2$ in $f_1(x)$ and $f_2(x)$. Show that your answer is correct.

2. Find the Fourier series for $f(x)$ defined on $(-1, 1)$ by

   $$f(x) = 0, \text{ for } -1 < x < 0 \qquad f(x) = 1, \text{ for } 0 < x < 1.$$

   What value does the series have at $x = 0$?

3. Evaluate the Fourier series for $f(\theta) = \theta^2$ on $(-\pi, \pi)$. Substitute $\theta = \pi/2$ and show that

   $$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots = \frac{\pi^2}{12}$$

A table of integrals useful for Fourier series.

$$\int \sin ax\, dx = -\tfrac{1}{a} \cos ax$$
$$\int x \sin ax\, dx = \tfrac{1}{a^2}[\sin ax - ax \cos ax]$$
$$\int x^2 \sin ax\, dx = \tfrac{1}{a^3}[2ax \sin ax - (a^2x^2 - 2) \cos ax]$$
$$\int e^{ax} \sin bx\, dx = \tfrac{e^{ax}[a \sin bx - b \cos bx]}{a^2+b^2}$$
$$\int \cos ax\, dx = \tfrac{1}{a} \sin ax$$
$$\int x \cos ax\, dx = \tfrac{1}{a^2}[\cos ax + ax \sin ax]$$
$$\int x^2 \cos ax\, dx = \tfrac{1}{a^3}[2ax \cos ax + (a^2x^2 - 2) \sin ax]$$
$$\int e^{ax} \cos bx\, dx = \tfrac{e^{ax}[a \cos bx + b \sin bx]}{a^2+b^2}$$
$$\int \sin^2 ax\, dx = \tfrac{1}{a}\left[\tfrac{ax}{2} - \tfrac{1}{4} \sin 2ax\right]$$
$$\int \cos^2 ax\, dx = \tfrac{1}{a}\left[\tfrac{ax}{2} + \tfrac{1}{4} \sin 2ax\right]$$
$$\int \sin mx \sin nx\, dx = \tfrac{\sin(m-n)x}{2(m-n)} - \tfrac{\sin(m+n)x}{2(m+n)} \qquad m^2 \neq n^2$$
$$\int \cos mx \cos nx\, dx = \tfrac{\sin(m-n)x}{2(m-n)} + \tfrac{\sin(m+n)x}{2(m+n)} \qquad m^2 \neq n^2$$
$$\int \cos mx \sin nx\, dx = \tfrac{\cos(m-n)x}{2(m-n)} - \tfrac{\cos(m+n)x}{2(m+n)} \qquad m^2 \neq n^2$$

Some special sines and cosines:

$$\sin n\pi = 0$$
$$\cos n\pi = (-1)^n$$
$$\sin \tfrac{n\pi}{2} = \begin{cases} 0 & n \text{ even} \\ (-1)^{(n-1)/2} & n \text{ odd} \end{cases}$$
$$\cos \tfrac{n\pi}{2} = \begin{cases} 0 & n \text{ odd} \\ (-1)^{n/2} & n \text{ even} \end{cases}$$