QM I: Examples

1. Find the Broglie wave length for the following

   1. a 70 kg man travelling at 6 km/h.
   2. a 1 g stone travelling at 10 m/s.
   3. a 10^{-6} g stone travelling at 1 m/s.

2. Try to repeat my "derivation" of the Schrödinger equation in three dimensions, where a plane wave is given by

$$e^{i\boldsymbol{k}\cdot\boldsymbol{r}-\omega t}\,.$$

   What is the differential realization of momentum?

3. Show that if $\psi_1(x,t)$ and $\psi_2(x,t)$ are both solutions of the time-dependent Schrödinger equation, $\psi_1(x,t)+\psi_2(x,t)$ is a solution as well.

4. Normalise the wave functions

$$\begin{aligned}\phi(x) &= x(l-x) \quad 0\leq x\leq l,\\ \phi(x) &= 0 \quad x\leq 0 \text{ and } x\geq l.\end{aligned}$$

   test.

5. State the continuity conditions for a solution to the time-independent Schrödinger equation for the potential

$$V(x)=\begin{cases}0 & |x|>a\\ -V_0 & -a<x<0\\ -2V_0 & 0<x<a\end{cases}\,.$$

   (You do not have to solve the resulting equations!)

6. Show that the solution for the square well potential, as discussed in class, are either even or odd, i.e.,

$$\phi(-x)=\pm\phi(x).$$

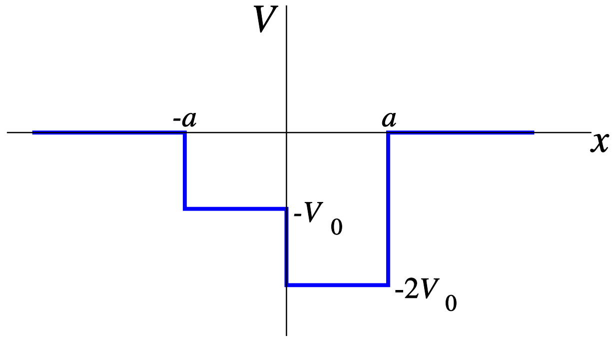

Figure 1: The potential

7. Show that the eigenfunctions of the infinitely deep square well are orthogonal. (Use the angle doubling formulas for trigoniometric functions, $\cos x\cos y=\frac{1}{2}(\cos(x+y)+\cos(x-y))$, $\sin x\cos y=\frac{1}{2}(\sin(x+y)+\sin(x-y))$ and $\sin x\sin y=\frac{1}{2}(-\cos(x+y)+\cos(x-y))$.)

8. Show that there is no bound-state solution for the square well for energy less than the depth of the well.

9. Explain the similarities and differences between the ground state wave functions of an infinite and a finite well.

10. Determine the reflection and transmission coefficient for the square well of depth V_0 (*not* the square barrier!) for $E>0$.

11. Show that the lowest four harmonic-oscillator eigenfunctions are orthogonal,

$$\begin{aligned}\phi_0(y) &= e^{-y^2/2},\\ \phi_1(y) &= ye^{-y^2/2},\\ \phi_2(y) &= (1-2y^2)e^{-y^2/2},\\ \phi_3(y) &= (y-\tfrac{2}{3}y^3)e^{-y^2/2}.\end{aligned}$$

   (Use the integral $\int_{-\infty}^{\infty}e^{-ay^2}dy=\sqrt{\pi}/\sqrt{a}$. Repeated differentiation w.r.t. a gives the integrals $\int y^{2n}e^{-y^2}$.)

12. A solution $\phi_1(r)$ of the time-independent Schrödinger equation at energy E_1 corresponds to a solution $\psi_1(r,t) = \phi_1(r)e^{-iE_1/\hbar t}$ of the time-dependent equation. What is the probability density when I superimpose two solutions, which are eigenstates for different energies, E_1 and E_2? (I.e., $\psi(r,t) = \psi_1(r,t) + \psi_2(r,t)$.) Analyse the time-dependent part of the probability density.

13. Calculate the expectation value of x and x^2 in the harmonic oscillator ground state

$$\phi_0(x) = \exp\left(-\frac{m\omega x^2}{2\hbar}\right).$$

Do the same for p and p^2. Define $(\Delta x)^2 =< x^2 > - < x >^2$, and similar for p. What is $\Delta x \Delta p$?

14. The uncertainty relation for position and momentum is $\Delta x \Delta p \gtrsim \hbar/2$. Use this formula to estimate the energy of the lowest state in an infinite square well. **Hint:** Assume $\Delta x = a$ (why is that a good assumption?). Use $< p^2 > \approx (\Delta p)^2$ (why?), and find the minimum value for the energy.

15. Show, using creation and annihilation operators, that

$$\begin{aligned} O_{nm} &= \int_{-\infty}^{\infty} u_n(y)^* u_m(y)\, dy \\ &\propto \int_{-\infty}^{\infty} \left[\left(\hat{a}^\dagger\right)^n e^{-y^2/2}\right]^* \left(\hat{a}^\dagger\right)^m e^{-y^2/2}\, dy \\ &= 0 \end{aligned}$$

if $n \neq m$.

16. A particle moves in an infinite well extending form $x = -a$ to $x = a$. At $t = 0$ its wave function is

$$\psi(x,0) = \frac{1}{\sqrt{5a}} \cos\left(\frac{\pi x}{2a}\right) + \frac{2}{\sqrt{5a}} \sin\left(\frac{\pi x}{a}\right).$$

What are the possible results of a measurement of the energy of this system and what are their relative probabilities? What is the full time dependent wave function?

17. A particle of mass m is in the lowest state of an infinite well extending from $x = -a$ to $x = a$. The width of the well is (somehow) suddenly expanded to twice its original size; so fast that the wave function does not change. Calculate the probabilities that the system is in (i) its new ground state (ii) the second state (iii) the third state.

18. A particle moves in a three-dimensional spherically symmetric well, specified by a potential $V(r)$, where $V(r) = -V_0$ for $r \leq a$, and $V(r) = 0$ for $r > a$. Show that the energies for states with quantum number $l = 0$ are determined by a relation of the form

$$\kappa a \cot \kappa a = -ka.$$

Find the smallest value of V_0 where a bound-state solution exists.

19. The energy of the deuteron (neutron+proton) is -2.23 MeV. Assuming that their interaction potential is of the form discussed in the previous question, and $a = 2.0\,10^{-15}$m, find the value of V_0.
Hint: the mass entering the Schrödinger equation is *half* the nucleon mass; $x = 1.82$ is a solution to $x \cot x = -0.46$.

20. A particle has a wave function of the form

$$z \exp(-\alpha(x^2 + y^2 + z^2)).$$

Show that this state is an eigenstate of the operators $\hat{L}^2$ and $\hat{L}_z$.
Hint: Either work with the explicit form of the operators in Cartesian coordinates, or express the wave function and operator in spherical coordinates.